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A STUDY OF EXTREMELY-LOW-FREQUENCY WAVE MOTIONS

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This report is concerned with a proper treatment of disturbances in the ionosphere. It develops the treatment of the excitation of extra-low-frequency (magneto-hydrodynamic-like) wave-motions in an infinite, homogeneous plasma imbedded in a constant, unidirectional magnetic field, for which the electrical conductivity tensor is easily established and methods of analysis are highly developed.

The problem treated in some detail is the radiation by charges traveling along the magnetic field with velocities comparable to the characteristic phase velocity (Alfven velocity) for low frequency waves. The problem resembles, to some extent, the ordinary Cerenkov problem in which charged particles, by a collective mechanism, generate visible radiation when passing through transparent substances. The ELF analogue -- which we call the magneto gas dynamic analogue of Cerenkov radiation -- has several distinguishing features. For one, the radiation is most important in a narrow cone of angles centered on the magnetic field line of the source. In addition, the magnetic field of the wave is transverse but the electric field is longitudinal. The disturbance is thus not circularly polarized as these are not transverse waves. Also, in addition to the usual Cerenkov component, for which the particle velocity must be greater than <u>a</u> (the Alfven phase velocity), there exists a weaker component originating from particles less than <u>a</u>.

The latter portion of the report represents a preparatory step in the development of a Microscopic Approach to plasma problems, by solving the first two equations in the BBGKY hierarchy in two noncorrelation limits. The possibility of deriving the dispersion relation for a sound wave propagating in a hard-sphere gas directly from the linearized Boltzmann equation for hard spheres, is also discussed.

17. INDEXING ANNOTATION

Treatment of the excitation of Extra-Low-Frequency wave motions in an infinite, homogeneous plasma in a constant, unidirectional magnetic field.

Microscopic approach to plasma problems.

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CHAPTER I

INTRODUCTION

The work reported here consists of two distinct efforts. The first effort, will generally be referred to by the heading "Excitation Problem". It aims at explaining the extra-low-frequency signals observed to occur naturally[1] and also in conjunction with high altitude nuclear explosions[2] in the ionized regions of the atmosphere. Only the simplest model ionosphere[3] is used in this treatment -- a uniform, infinite, homogeneous two-fluid plasma, neutral overall, embedded in a constant uniform magnetic field; only Coulomb and electromagnetic forces are considered; only the linearized average field (Vlasov) approximation is used to determine the dielectric and conductive behavior of the plasma; and collisions are entirely ignored.

The second effort aims at improving the description of the plasma. This is done by treating the inter-particle forces in a higher approximation, the result of which is to bring in collisions and in other ways modify the dielectric and conductive properties of the plasma. The modes of oscillation and propagation which then exist for the plasma in interaction with the electromagnetic field are expected to provide new insights into plasma behavior, and possibly also into ionospheric phenomena, which are very diverse and little understood. Because the microscopic treatment of the plasma is the only feasible one in this endeavor we shall refer to the second effort by the heading "Microscopic Approach."

Recent experimental and theoretical progress in geophysics has greatly clarified the gross features of the ionosphere and the exosphere. This progress has come about largely through direct exploration of the earth's neighborhood by instrumented satellites, and high-altitude nuclear and thermo-nuclear explosion experiments. The satellite experiments measure the normal features of the several ionized regions, together with naturally occurring disturbances in these regions. The explosion experiments measure the dynamical-response properties of these regions, and are therefore complementary to the others. The picture of the earth's ionized atmosphere which emerges illustrates the competition between the influence of the earth's magnetic field on the charged particles entering it and the particles' influence on the earth's field. Where the particle densities are relatively small, (such as in the Van Allen regions and in the artifically created electron shells), the charged particles follow the characteristic Stormer orbits. However, where, the earth magnetic field is sufficiently weak the field lines get dragged along (so to speak) by the particle fluxes (as far as we know entirely from the sun). The region around the earth which (apart from disturbances) is dominated by and moves with the earth's magnetic field is often referred to as the magnetosphere. Because the axis of the earth's dipole field lines cross-wise to the plasma streaming from the sun -- (referred to as the solar wind), the earth is effectively shielded within its magnetosphere. Charges are deflected away from the earth, or trapped in the field, or focussed towards the polar regions. As a consequence, not all of the magnetic disturbances seen outside the magnetosphere lead to magnetic disturbances at the earth's surface.

The magnetosphere appears to be typically some 8-10 earth-radii (Re) distant (from the center of the earth) on the sunward side, and perhaps 20-30 Re distant on the opposite side of the earth. Within these limits we may properly take the earth's magnetic field to be constant, of dipole shape, and unaffected by the ionized atmosphere which is distributed (somewhat unevenly) throughout it. A proper treatment of disturbances in the magnetosphere -- which is the object of the present work -- is hardly possible by analytical methods if one wants to include the dipole nature of the ambient magnetic field as well as the variations in particle densities and particle flux densities. This is particularly true when one wants to study oscillatory and wave-like phenomena at such low frequencies that the relevant wave-lengths are comparable to or over-shadow the structure and spatial variations of the magnetic field and particle distributions. However, so much remains to be learned of a qualitative and semi-quantitative nature about the interaction of particles (natural and artificial) with the magnetosphere, and about the propagation of disturbances generated within it and at its surface, as to justify the study of more tractable situations. In particular, in the present study, we develop the treatment of the excitation of extra-low-frequency (magnetohydrodynamic-like) wave motions in an infinite homogeneous plasma imbedded in a constant, unidirectional magnetic field for which the electrical conductivity tensor is easily established and methods of analysis are highly developed.

The problem we treat in some detail is the radiation by charges traveling along the magnetic field with velocities comparable to the characteristic phase velocity (Alfven velocity) for low frequency waves. The problem resembles, to some extent, the ordinary Cerenkov problem,[4] in which charged particles, by a collective mechanism, generate visible radiation when passing through transparent substances. The ELF analogue -- which we call the magneto-gas-dynamic analogue of Cerenkov radiation -- has several distinguishing features. For one, the radiation is most important in a narrow cone of angles centered on the magnetic field line of the source. For another, the magnetic field of the wave is transverse, but the electric field is longitudinal. The disturbance is not circularly polarized, (i.e., these are not transverse waves). In addition, besides the usual Cerenkov component for which v, the particle velocity, must be numerically greater than a, the Alfven phase velocity, there exists a weaker component originating from particles of velocity less than a.

We believe that some of the naturally occurring low frequency disturbances noted on magnetogram records -- especially those from observatories located in the more northerly, magnetically active areas -- can be understood in terms of the Cerenkov mechanism with reasonable estimates of the particle fluxes even though at low frequencies the effect is in general very weak. Possibly of even greater promise is the application of our calculations to the hydromagnetic signals definitely known to accompany high altitude nuclear explosions. Both of these applications will be presented in the sequel to this report.

Chapter II is devoted to the development of the excitation formulation, a development which proceeds naturally from an earlier report by Cantor, Keilson and Schneider[3] that considered a non-local (temperature-dependent) model of the ionosphere. The formulation given there is carried over and brought to a useful stage. That report and this are both motivated by a desire to found the theory of ionized gases on a secure microscopic (kinetic) basis, to set aside the doubtful macroscopic equations in favor of less doubtful 'molecular' equations that properly include the interparticle forces. For the problem treated -- the zero-temperature collisionless plasma -- there is no practical advantage to the philosophy; the results of the two approaches are identical as far as the electromagnetic response functions are concerned. This is due to the fact that in the earlier report the basic kinetic approximation was the average-field or Vlasov approximation, that treats each particle as if it moved in the average field of all others. The consequence for the zero temperature plasma (and also the low-temperature plasma) is that the molecular encounters[5] -- the true sign of 'kinetic' behavior -- were neglected. The later sections of this report explore the possibility of bringing a more refined description into the theory (as an improvement in the electromagnetic response function).

In Chapter III the linear electric field response to an impulse-current source is computed in some detail for the frequency range below the ion cyclotron frequency.[6] This "radiation Green's Function" is the simplest function to illustrate the directional properties of Alfven-like waves generated in a compressible conducting fluid. The coupling of electromagnetic and kinetic processes is by way of the Vlasov average-field approximation.

The radiation Green's function is an isotropic matrix of functions exhibiting five distinct kinds of response to a point excitation. First, there is the excited field. This occurs even in the absence of an external magnetic field. It exhibits isotropic propagation for those components whose frequencies are above the plasma frequency, and damping for frequencies below the plasma frequency. Second, there is a signal which is propagated at the Alfven phase-velocity equally in all directions. This signal is characteristic of the compressible fluid. Third, there is a signal which is propagated inside a double-cone centered on a magnetic field line with apex at the source point, and which is damped outside of the cone. The narrowness of the cone varies with the emitted frequency, becoming most narrow at the lowest frequencies. It is this which corresponds to the Alfven mode (propagation along the magnetic field) observed in incompressible conducting fluids. The modification due to compressibility is such as to make this part of the radiation field singular on the surface of the cone. The fourth and fifth kinds of behavior are caused by the interference or competition of the Alfven-mode with each of the first and second kinds of behavior. When the interference is with the (first) damped mode, the propagation in the cone is lost. When the interference is with the (second) isotropically propagating mode, the cone boundary becomes diffuse and the radiation is enhanced, especially along the magnetic field direction.

The Normal Cerenkov Effect is examined in Chapter IV, and several of the mathematical details that will be needed to calculate the Magneto-Gas-Dynamic Analogue of the Cerenkov Effect are presented. This calculation -- the main effort of the report -- is given in Chapter V. We have already stated some of the ways in which the Magneto-Effect differs from the Normal Effect. In both cases the radiation appears only at a specific angle with respect to the motion of the charge (same as the direction of the ambient magnetic field). This angle depends only on the velocity (v) of the charge relative to the Alfven velocity (a). But whereas normally v must be greater than a, now it may also be less than a, though this "anomalous" effect is negligibly small compared to the main effect. (However, it has important theoretical consequences.) The radiation is, in both cases, linearly polarized, but unlike the normal-effect the electric field of the radiation is only along the ambient magnetic field (B_o) while the magnetic field of the radiation is only perpendicular to it. The "radiation," therefore, is in a "hydromagnetic" mode. Furthermore, the rate of radiation, as a function of angle, is greatest near the magnetic field direction, when v is very close to but slightly different from a. However, it goes to zero when v = a. The latter behavior is also true of the normal Cerenkov radiation, but in the magnetic case it comes about through a cancellation of two terms which, when v is not too close to a, give rise separately to the "main" and "anomolous" effects.

Chapter VI is given over to a discussion of the temperature corrections that may be expected to play a part in ELF propagation, and they are found not to be important in the examples we have considered, though this is by no means always the case. More will be said on this point in the sequel.

The final two Chapters represent a preparatory step in the development of a Microscopic Approach to plasma problems. Chapter VII shows how, in terms of a sequence of distribution function $f^{(1)}(rvt)$, $f^{(2)}(rv,r'v't)$, etc., describing the average properties of an assembly of particles in successively greater detail, one can formulate the microscopic laws[7] of the system. The resulting equations are solved in two non-correlation limits -- the first leads to the Vlasov approximation; the second leads to the (generalized) Boltzmann Equation, which includes the Vlasov limit. To illustrate its relation to the more customary Boltzmann Equation we apply it to the case of hard-spheres, actually using a limiting form of interaction potential to describe collisions. In the sequel we shall apply it to the two fluid plasma with special attention to the rederivation of the plasma conductivity. In Chapter VIII we discuss the possibility of deriving directly the dispersion relation for a sound wave in the hard-sphere gas rising the linearized Boltzmann Equation for hard spheres, and find that the difficulties are great, even in the one-dimensional approximation (which has been treated with some success by other[8] methods).

Summary and Outlook

To summarize: an improved understanding of ionospheric (and other) plasma phenomena requires (a) detailed calculations of the electromagnetic response to realistic excitations and (b) development of methods for including in the plasma description other consequences of the collective behavior than just the average-field, space-charge forces. With regard to (a) we may say that although there has been a great deal of attention to the magneto-ionic modes, the more difficult, and tedious, excitation problems have been generally sidestepped. In the present work we tackle the problem of ELF wave-motions generated by charges moving along the magnetic field, and in the sequel we shall include an additional spiralling motion to duplicate the actual behavior of charges injected into the earth's field without preferential direction.

With regard to (b) we can say that our aim is to include, within the framework of a general formalism, the modifications of the electrical susceptibility and conductivity tensors which are a consequence of the "higher correlations" of particle behavior. The most important of such modifications will, for our purposes, most likely be those due to internal collisions. For the ionosphere, another kind of collision is important -- external collision, i.e., of electrons and ions with the neutral molecules of the gas. One of our motivations in treating the hard-sphere gas is to provide some idea for the analytical treatment of external collisions. In earlier work[3] a Fokker-Planck model was used for this purpose, but even though it had several nice features it led to no practical improvement over the simpler methods used in the Appleton-Hartree theories of ionospheric conductivity of a cold plasma.

Finally, in the next report we shall present the application of the results of our calculations to both natural ionospheric ELF phenomena, and to the artifically created ELF signals of the various high-altitude nuclear explosion experiments.

CHAPTER II

FORMULATION OF THE EXCITATION PROBLEM

In line with our program to deduce the macroscopic plasma behavior from molecular laws, we shall write Maxwell's equation for vacuum processes and deduce the dielectric and conductive parameters from the net response of the charges and currents in the vacuum to applied fields.

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{2.1}$$

$$\nabla \cdot E = \rho/\varepsilon_o \tag{2.2}$$

$$\nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t} + \mu_o j \tag{2.3}$$

$$\nabla \cdot B = 0 \quad . \tag{2.4}$$

Throughout this work we shall employ an implicit vector notation. No special mark will designate a vector, and it will be understood that the electric field E, the magnetic field B, the electric current density j, and the gradient operation ∇ are vectors, while the electric charge density ρ is a scalar and the time-derivative $\frac{\partial}{\partial t}$ is a scalar operation. Furthermore, ρ and the separate components of E, B and j are functions of the vector position $r = (x,y,z)$ and the time t. The "dot" denotes scalar product, the "cross" vector product.

The constants, $\mu_o = 4\pi \times 10^{-7}$ henries/meter and $\varepsilon_o \cong 1/36\pi \times 10^{-9}$ farads/meter, are related by the velocity of light ($c = 3 \times 10^8$ meters/se according to

$$\varepsilon_o\mu_o = 1/c^2 \quad . \tag{2.5}$$

Consequently, charge conservation follows from the above:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \underset{\sim}{j} = 0 \quad . \tag{2.6}$$

Finally, the Lorentz force (a vecotr F) on a particle of charge q, velocity v (a vector), at rt (space-time coordinates) takes the form:

$$\underset{\sim}{F} = q(\underset{\sim}{E}(\underset{\sim}{r}t) + \underset{\sim}{v} \times \underset{\sim}{B}(\underset{\sim}{r}t)) \quad . \tag{2.7}$$

[The applied fields will be a constant magnetic field and the field of a prescribed charge and current source, or as we shall call it an external source.] That part of ρ and j which is not prescribed will necessarily have been induced in the plasma:

$$\rho = \rho^{ext} + \rho^{ind} \tag{2.8}$$

$$\underset{\sim}{j} = \underset{\sim}{j}^{ext} + \underset{\sim}{j}^{ind} \tag{2.9}$$

The induced charge and current densities are due to the motion of the constituent particles, the equations of motion for which contain the <u>total fields</u> $\underset{\sim}{E}$ and $\underset{\sim}{B}$.

The plasma we have in mind is, in its undisturbed state, uniform, homogeneous, and in equilibrium. The presence of a constant magnetic field disturbs only the isotropy of the plasma's response, but not its homogeneity or equilibrium. This term is characterized by no net charges or currents, no electric fields, and no magnetic fields other than B_o. There will be fluctuations of these quantities because of the molecular nature of ions and electrons, but we shall ignore these <u>fluctuations</u> relative to the <u>disturbances</u> which we do treat. For our purposes the plasma will be described by entirely continuous, slowly varying functions (slow relative to the fluctuations).

A disturbance of the plasma will be characterized by a non-vanishing E and B, together with accompanying charge and current densities. We should suppose that, in the sense of Ohm's law, there is a linear relation between ρ^{ind}, $\underset{\sim}{j}^{ind}$ and $\underset{\sim}{E}$, $\underset{\sim}{B}$. We shall discover that so long as the only source of anisotropy is $\underset{\sim}{B}_o$, only $\underset{\sim}{E}$ enters these relations. They are:[3]

$$\rho^{ind}(\underset{\sim}{r}t) = -\nabla\cdot\varepsilon_o \int_{-\infty}^{\infty} (d^3\underset{\sim}{r}') \int_{-\infty}^{\infty} dt' \; X(\underset{\sim}{r}-\underset{\sim}{r}', t-t')\cdot\underset{\sim}{E}(\underset{\sim}{r}'t') \tag{2.10}$$

$$\underset{\sim}{j}^{ind}(\underset{\sim}{r}t) = \int_{-\infty}^{\infty} (d^3\underset{\sim}{r}') \int_{-\infty}^{\infty} dt' \; \sigma(\underset{\sim}{r}-\underset{\sim}{r}', t-t')\cdot E(r't') \quad . \tag{2-11}$$

The quantities X (susceptibility tensor) and σ (conductivity tensor) will be understood to be tensors, and dyadic notation will be employed. If we use $i,j,k,\cdots = 1,2,3$ to distinguish the components of the vectors and tensors; then in more detail we can write:

$$\rho^{ind}(\underset{\sim}{r}t) = -\sum_{ij} \nabla_i \int_{r'} (d^3\underset{\sim}{r}') \int_{t'} dt' \; X_{ij}(\underset{\sim}{r}-\underset{\sim}{r}', t-t')E_j(\underset{\sim}{r}'t') \tag{2.12}$$

$$\underset{\sim}{j}_i^{ind}(\underset{\sim}{r}t) = \sum_j \int_{r'} (d^3\underset{\sim}{r}') \int_{t'} dt' \; \sigma_{ij}(\underset{\sim}{r}-\underset{\sim}{r}', t-t') \; E_j(\underset{\sim}{r}'t') \quad . \tag{2.13}$$

These are non-local relations: ρ and j at one place and time depend on E for all other places and times. Actually, microscopic causality implies that only fields at earlier times determine ρ and j. We may express this by the demand

$$\overset{\Rightarrow}{X}(t-t') = 0 \qquad t < t' \tag{2.14}$$

$$\overset{\Rightarrow}{\sigma}(t-t') = 0 \qquad t < t' \tag{2.15}$$

and continue to extend all the integrals from $-\infty$ to $+\infty$. Thus the induced charge and current are given by fourfold convolutions of the electric field with tensors characterizing the plasma's responsiveness. That X and σ are functions only of $r-r'$ and $t-t'$ is true because the equilibrium state is independent of r and t. It is especially convenient, in such cases, to use Fourier integral transforms since the transform of a convolution is simply the product of the transforms of the factors.

For notation we shall use the same symbol for the transform as for the original function, and shall use a standard convention:

$$f(\underset{\sim}{r}t) = \int_{-\infty}^{\infty} \frac{(d^3\underset{\sim}{k})}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}-i\omega t} f(\underset{\sim}{k}\omega) \tag{2.16}$$

Then

$$\rho^{ind}(\underset{\sim}{k}\omega) = -i\underset{\sim}{k}\cdot\varepsilon_o\overset{\rightrightarrows}{X}(\underset{\sim}{k}\omega)\cdot E(\underset{\sim}{k}\omega) \tag{2.17}$$

$$\underset{\sim}{j}^{ind}(\underset{\sim}{k}\omega) = \sigma(\underset{\sim}{k}\omega)\cdot\underset{\sim}{E}(\underset{\sim}{k}\omega) \quad . \tag{2.18}$$

As we may say, these relations are "local relations in $\underset{\sim}{k}$-ω-space." In ordinary parlance, ω is the (angular) wave frequency and $\underset{\sim}{k}$ the wave-vector of the $k\omega$ component of the electric field.

From conservation of charge there emerges an important relation between $\overset{\rightrightarrows}{X}$ and $\overset{\rightrightarrows}{\sigma}$. Since ρ^{ext}, $\underset{\sim}{j}^{ext}$ separately obey the conservation equation, ρ^{ind}, $\underset{\sim}{j}^{ind}$ do so too. Substituting (2.10) and (2.11) into (2.6), we realize that there is sufficient arbitraries in $\underset{\sim}{E}(\underset{\sim}{r}t)$ to conclude that

$$\nabla_{\underset{\sim}{r}}\cdot\overset{\rightrightarrows}{\sigma}(\underset{\sim}{r}-\underset{\sim}{r}',t-t') = \frac{\partial}{\partial t}\nabla_{\underset{\sim}{r}}\cdot\varepsilon_o\overset{\rightrightarrows}{X}(\underset{\sim}{r}-\underset{\sim}{r}',t-t') \tag{2.19}$$

or

$$\underset{\sim}{k}\cdot\overleftrightarrow{\sigma}(\underset{\sim}{k}\omega) = -i\varepsilon_o\omega\underset{\sim}{k}\cdot\overleftrightarrow{X}(\underset{\sim}{k}\omega) \tag{2.20}$$

in the everpresent dyadic notation.

From Maxwell's Equation we deduce the following second-order equation:

$$(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\,\underset{\sim}{E} = -\frac{1}{\varepsilon_o}\nabla(\rho^{ext} + \rho^{ind}) = \mu_o\frac{\partial}{\partial t}(\underset{\sim}{j}^{ext} + \underset{\sim}{j}^{ind}) \tag{2.21}$$

or,

$$(k^2 - \frac{\omega^2}{c^2})\underset{\sim}{E} = -\frac{i\underset{\sim}{k}}{\varepsilon_o}(\rho^{ext}(\underset{\sim}{k}\omega) + \rho^{ind}(\underset{\sim}{k}\omega)) + i\omega\mu_o(\underset{\sim}{j}^{ext}(\underset{\sim}{k}\omega) + \underset{\sim}{j}^{ind}(\underset{\sim}{k}\omega)) \tag{2.22}$$

or,

$$(k^2 - \frac{\omega^2}{c^2})\underset{\sim}{E} + \underset{\sim}{k}(\underset{\sim}{k}\cdot\overleftrightarrow{X}\cdot\underset{\sim}{E}) - i\mu_o\omega\overleftrightarrow{\sigma}\cdot\underset{\sim}{E} = -i(\underset{\sim}{k}\,\frac{\rho^{ext}}{\varepsilon_o} - \mu_o\omega\underset{\sim}{j}^{ext} \tag{2.23}$$

making use of (2.17) and (2.18). Because of the conservation laws, $\underset{\sim}{j}^{ext}$ contains whatever significant information is in ρ^{ext}. (The converse is not true.) An equation containing $\underset{\sim}{j}^{ext}$ only is remarkably symmetrical. Multiply (2.23) by $\underset{\sim}{k}\cdot$, and factor out $(k^2 - \omega^2/c^2)$ after using (2.6) and (2.20):

$$\underset{\sim}{k}\cdot\underset{\sim}{E} + \underset{\sim}{k}\cdot\overleftrightarrow{X}\cdot\underset{\sim}{E} = -\,i\,\rho^{ext}/\varepsilon_o \quad . \tag{2.24}$$

Hence upon eliminating ρ^{ext}, we get

$$(k^2 - \frac{\omega^2}{c^2})\underline{E} - \underline{k}(\underline{k}\cdot\underline{E}) - i\mu_o\omega\overleftrightarrow{\sigma}\cdot\underline{E} = i\mu_o\omega\underline{j}^{ext} \quad . \tag{2.25}$$

Let us introduce the tensor $\hat{X}$ by the relation

$$\overleftrightarrow{\sigma} = - i\varepsilon_o\omega\overleftrightarrow{\hat{X}} \quad . \tag{2.26}$$

Then

$$\left[(k^2 - \frac{\omega^2}{c^2})\overleftrightarrow{1} - \underline{k}\underline{k} - \frac{\omega^2}{c^2}\overleftrightarrow{\hat{X}}\right]\cdot\underline{E} = i\mu_o\omega\underline{j}^{ext} \tag{2.27}$$

and

$$\underline{k}\cdot\overleftrightarrow{\hat{X}} = \underline{k}\cdot\overleftrightarrow{X} \quad . \tag{2.28}$$

The symbol $\overleftrightarrow{1}$ is the unit dyadic; $\overleftrightarrow{X}$ is a dyadic; $\overleftrightarrow{1}\cdot\underline{E} \equiv \underline{E}, \underline{k}\underline{k}\cdot\underline{E} \equiv \underline{k}(\underline{k}\cdot\underline{E})$.

The quantity in brackets in (2.27) is, in Cartesian components, a matrix, whose inverse allows us to solve for $\underline{E}$. The result, when transformed back to $\underline{r}t$-space is the field excited by the current $\underline{j}^{ext}$. The formal solution is

$$\underline{E}(\underline{k}\omega) = i\mu_o\omega \frac{\overleftrightarrow{1}}{(k^2 - \frac{\omega^2}{c^2})\overleftrightarrow{1} - \underline{k}\underline{k} - \frac{\omega^2}{c^2}\overleftrightarrow{\hat{X}}(k\omega)} \underline{j}^{ext}(\underline{k}\omega) \quad . \tag{2.29}$$

Let us evaluate the matrix before $\underline{j}^{ext}$.

Introduce the scalars

$$\beta = \frac{1}{k^2 - \omega^2/c^2} \tag{2.30}$$

and

$$\alpha = \frac{\omega^2/c^2}{k^2 - \omega^2/c^2} \quad . \tag{2.31}$$

Then

$$\overleftrightarrow{A} = \frac{1}{(k^2 - \frac{\omega^2}{c^2})\,\overleftrightarrow{1} - \underset{\sim}{k}\underset{\sim}{k} - \frac{\omega^2}{c^2}\,\overleftrightarrow{X}} = \frac{\beta\overleftrightarrow{1}}{\overleftrightarrow{1} - \beta\underset{\sim}{k}\underset{\sim}{k} - \alpha\,\overleftrightarrow{X}} \quad . \tag{2.32}$$

We first treat $1 - \alpha\hat{X}$ as a unit, and formally expand the denominator in powers of β:

$$\overleftrightarrow{A} = \beta\left[\frac{\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\overleftrightarrow{X}} + \frac{\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\overleftrightarrow{X}} \cdot \beta\underset{\sim}{k}\underset{\sim}{k} \cdot \frac{\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\overleftrightarrow{X}}\right.$$

$$\left. + \frac{\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\hat{X}} \cdot \beta\underset{\sim}{k}\underset{\sim}{k} \cdot \frac{\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\hat{X}} \cdot \beta\underset{\sim}{k}\underset{\sim}{k} \cdot \frac{\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\hat{X}} + \cdots\right] . \tag{2.33}$$

The first term is irreducible; the second term is the outer product of two vectors; the third term is like the second, except for a scalar factor; each higher term contains another identical scalar factor. $\overleftrightarrow{A}$ can therefore be written as two terms, by summing the geometrical series of scalar factors.

$$\overleftrightarrow{A} = \frac{\beta\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\overleftrightarrow{X}} + \frac{\beta\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\overleftrightarrow{X}} \cdot \underset{\sim}{k}\, \frac{1}{1 - (\underset{\sim}{k} \cdot \frac{\beta\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\overleftrightarrow{X}} \cdot \underset{\sim}{k})}\, \underset{\sim}{k} \cdot \frac{\beta\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\overleftrightarrow{X}} \quad . \tag{2.34}$$

It has been necessary to use the same symbol 1 for the unit matrix and the scalar unity. The latter appears only in the large parenthesis, which is the sum of the series of scalars:

$$1 + (\underset{\sim}{k} \cdot \frac{\beta\overleftrightarrow{1}}{\overleftrightarrow{1} - \alpha\overleftrightarrow{X}} \cdot \underset{\sim}{k}) + (\qquad)^2 + (\qquad)^3 + \cdots$$

We shall introduce the symbol

$$\overset{\rightrightarrows}{M} = \frac{\beta\overset{\rightrightarrows}{1}}{\overset{\rightrightarrows}{1} - \alpha\overset{\rightrightarrows}{X}} = (\beta \text{ times inverse of } \overset{\rightrightarrows}{1} - \alpha\overset{\rightrightarrows}{X}) \tag{2.35}$$

for the important matrix that appears four times.

$$\overset{\rightrightarrows}{A} = \overset{\rightrightarrows}{M} + \frac{\overset{\rightrightarrows}{M}\cdot\underset{\sim}{k}\underset{\sim}{k}\cdot\overset{\rightrightarrows}{M}}{1 - \underset{\sim}{k}\cdot\overset{\rightrightarrows}{M}\cdot\underset{\sim}{k}} \quad . \tag{2.36}$$

(The denominator is a scalar, so we needn't worry about its position in the formula. But we shall see to it that vectors and matrices are arranged according to dyadic notation. It will then always be clear how to convert them to component form.)

To proceed further will require a knowledge of $\overset{\rightrightarrows}{X}$, i.e., essentially the tensor conductivity of the plasma in its ambient magnetic field. So long as the plasma is at zero temperature and contains no drifting parts, $\overset{\rightrightarrows}{X}$ is identical to $\overset{\rightrightarrows}{X}$. That is, the conductivity and susceptibility tensors contain identical information. Referring back to (2.26), this means that

$$\overset{\rightrightarrows}{\sigma} = - i\varepsilon_o\omega\overset{\rightrightarrows}{X} \tag{2.37}$$

which is a stronger condition than charge conservation (2.20).

When we ask for the low frequency waves and oscillations excited by external currents, $\overset{\rightrightarrows}{X}$ simplifies immensely. Let us write (2.29) in the form

$$\underset{\sim}{E}(\underset{\sim}{k}\omega) = i\mu_o\omega\overset{\rightrightarrows}{A}(\underset{\sim}{k}\omega) \cdot \underset{\sim}{j}^{ext}(\underset{\sim}{k}\omega) \quad . \tag{2.38}$$

If $\underset{\sim}{j}^{ext}$ contains only low frequencies, $\underset{\sim}{E}$ will, in fact, contain the same low frequencies. What this means is that only $\overset{\rightrightarrows}{A}$'s corresponding to these low frequencies are relevant for the solution $\underset{\sim}{E}$. Hence, if we anticipate the later use of either low-frequency excitations or the low-frequency components of $\underset{\sim}{E}$ and $\underset{\sim}{B}$, it will suffice to simplify $\overleftrightarrow{A}$ according to its frequency dependence. The frequency range we have in mind is below the ion cyclotron frequencies, but above collision frequencies. This is the Alfven, or magneto-hydrodynamic, range, and is distinguished by one-dimensional wave propagation along the magnetic field lines. $\overleftrightarrow{X}$ must necessarily be highly anisotropic to achieve such a state of affairs. But the simplifying feature in this extra low-frequency region is that $\overleftrightarrow{X}$ is diagonal. Hence, $\overleftrightarrow{1} - \mu\overleftrightarrow{X}$ is diagonal, and $\overleftrightarrow{M}$ is diagonal: simply the matrix of reciprocals.

In a Cartesian coordinate system, let $\hat{z}$ be the magnetic field direction. Then the non-vanishing components will be[3],[9]

$$X_{xx} = X_{yy} = \sum_i \frac{\omega_{pi}^2}{\Omega_i^2} \tag{2.39}$$

$$X_{zz} = -\frac{\sum_i \omega_{pi}^2}{\omega^2} \tag{2.40}$$

The sum is over the various components of the plasma; for example, electrons and H^+ ions. ω_p is the plasma frequency, one for each component; Ω is the gyro-frequency, one of each component. We shall let the total plasma frequency be ω_p:

$$\omega_p^2 = \sum_i \omega_{pi}^2 \ . \tag{2.41}$$

It is approximately the same as the electron plasma frequency. Now

$$M_{xx} = M_{yy} = \frac{\beta}{1 - \alpha X_{xx}} \tag{2.42}$$

$$M_{zz} = \frac{\beta}{1 - \alpha X_{zz}} \quad . \tag{2.43}$$

The Alven velocity appears in M_{xx}; it is designated by the symbol a:

$$a^2 = \frac{c^2}{1 + \sum_i \left(\frac{\omega_p^2}{\Omega^2}\right)_i} \quad . \tag{2.44}$$

Hence,

$$M_T \equiv M_{xx} = \frac{1}{k^2 - \omega^2/a^2} \tag{2.45}$$

$$M_z \equiv M_{zz} = \frac{1}{k^2 + \omega_p^2/c^2 - \omega^2/c^2} \tag{2.46}$$

In this special case, $\overleftrightarrow{M}\cdot\underset{\sim}{k} = \underset{\sim}{k}\cdot\overleftrightarrow{M}$, a vector with the components

$$\underset{\sim}{k}\cdot\overleftrightarrow{M} = (k_x M_T, k_y M_T, k_z M_z) \quad . \tag{2.47}$$

Further,

$$\underset{\sim}{k}\cdot\overleftrightarrow{M}\cdot\underset{\sim}{k} = (k_x^2 + k_y^2) M_T + k_z^2 M_z \quad . \tag{2.48}$$

We can let k_T be that part of the vector k which is transverse to the magnetic field lines:

$$k_T = (k_x, k_y, 0) \qquad k_T^2 = k_x^2 + k_y^2 \tag{2.49}$$

Then

$$D = 1 - \underset{\sim}{k}\cdot\overset{\rightleftarrows}{M}\cdot\underset{\sim}{k} = 1 - \frac{k_T^2}{k^2 - \omega^2/a^2} - \frac{k_z^2}{k^2 + \omega_p^2/c^2 - \omega^2/c^2}$$

$$= \left(\frac{\omega_p^2}{c^2} - \frac{\omega^2}{c^2}\right)\left[k_z^2 - \frac{\omega^2}{a^2}\left(1 + \frac{k_T^2}{\frac{\omega_p^2}{c^2} - \frac{\omega^2}{c^2}}\right)\right] M_T M_z \qquad (2.50)$$

CHAPTER III

THE RADIATION GREEN'S FUNCTIONS

We now turn to the problem of determining the radiation fields caused by a single charged particle passing through the ionosphere.

The eye, or any other receiving instrument, is sensitive to the frequency components of the resultant radiation. The frequency components of the electric field at a point $\underset{\sim}{r}$, due to an excitation at a point $\underset{\sim}{r}'$, are written as a convolution integral

$$\underset{\sim}{E}(\underset{\sim}{r},\omega) = i_o\omega \int \overset{\rightleftarrows}{A}(\underset{\sim}{r}-\underset{\sim}{r}',\omega) \cdot \underset{\sim}{j}^{ext}(\underset{\sim}{r}',\omega)\, d^3\underset{\sim}{r}' \qquad (3.1)$$

where $\overset{\rightleftarrows}{A}$ is the matrix element defined in Chapter II, which appears explicitly in Equation (2.36).

Our immediate task is to determine $\overset{\rightleftarrows}{A}(\underset{\sim}{r}-\underset{\sim}{r}',\omega)$. The elements of A in wavenumber $(\underset{\sim}{k})$ space, $\overset{\rightleftarrows}{A}(\underset{\sim}{k},\omega)$ can be obtained directly, as was shown in Chapter II, Equation (2.30).

Then, the elements of the matrix $\overset{\rightleftarrows}{A}$ may be written

$$A_{ik} = M_{ik} + \frac{1}{D}\,(M_{ij}\; k_j\; k_\ell M_{\ell k}) \qquad (3.2)$$

where

$$D = \frac{\omega_p^2}{c^2}\left[k_z^2 - \frac{\omega^2}{a^2}\,(1 + c^2\,\frac{k_T^2}{\omega_p^2})\right] M_T M_z \qquad (3.3)$$

is the limit of Equation (2.50) when $\omega << \omega_p$. Throughout this paper, we have neglected ω^2/c^2 but not ω^2/a^2 with respect to ω_p^2/c^2. Explicitly

$$A_{xx} = M_T + \frac{1}{D} k_x^2 M_T^2 \tag{3.4}$$

$$A_{yy} = M_T + \frac{1}{D} M_T^2 k_y^2 \tag{3.5}$$

$$A_{zz} = M_z + \frac{1}{D} M_z^2 k_z^2 \tag{3.6}$$

$$A_{xy} = A_{yx} = \frac{1}{D} M_T^2 k_x k_y \tag{3.7}$$

$$A_{xz} = A_{zx} = \frac{1}{D} M_T M_z k_x k_z \tag{3.8}$$

$$A_{yz} = A_{zy} = \frac{1}{D} M_T M_z k_y k_z \tag{3.9}$$

It is readily seen from Equation (3.2) and from the nature of $\overleftrightarrow{M}$, that the tensor $\overleftrightarrow{A}$ is symmetric.

Substituting in (3.4) - (3.8) the expressions for M_T, M_z and D, we obtain

$$A_{xx} = \frac{1}{k^2 - \omega^2/a^2} + \frac{c^2}{\omega_p^2} \frac{k_x^2}{k_z^2 - \frac{\omega^2}{a^2}\left(1 + c^2 \frac{k_T^2}{\omega_p^2}\right)} \left[1 - \frac{(\omega^2/a^2 + \omega_p^2/c^2)}{k^2 + \omega_p^2/c^2}\right] \tag{3.10}$$

$$A_{yy} = \frac{1}{k^2 - \omega^2/a^2} + \frac{c^2}{\omega_p^2} \frac{k_y^2}{k_z^2 - \frac{\omega^2}{a^2}\left(1 + c^2 \frac{k_T^2}{\omega_p^2}\right)} \left[1 - \frac{(\omega^2/a^2 + \omega_p^2/c^2)}{k^2 + \omega_p^2/c^2}\right] \tag{3.11}$$

$$A_{zz} = \frac{1}{k^2 + \omega_p^2/c^2} + \frac{c^2}{\omega_p^2} \frac{k_z^2}{k_z^2 - \frac{\omega^2}{a^2}\left(1 + c^2 \frac{k_T^2}{\omega_p^2}\right)} \left[1 + \frac{(\omega^2/a^2 + \omega_p^2/c^2)}{k^2 - \omega^2/a^2}\right] \tag{3.12}$$

$$A_{xy} = A_{yx} = \frac{c^2}{\omega_p^2} \frac{k_x k_y}{k_z^2 - \frac{\omega^2}{a^2}\left(1 + c^2 \frac{k_T^2}{\omega_p^2}\right)} \left[1 - \frac{(\omega^2/c^2 + \omega_p^2/c^2)}{k^2 + \omega_p^2/c^2}\right] \tag{3.13}$$

$$A_{xz} = A_{zx} = \frac{c^2}{\omega_p^2} \frac{k_x k_z}{k_z^2 - \frac{\omega^2}{a^2}\left(1 + c^2 \frac{k_T^2}{\omega_p^2}\right)} \tag{3.14}$$

$$A_{yz} = A_{zy} = \frac{c^2}{\omega_p^2} \frac{k_y k_z}{k_z^2 - \frac{\omega^2}{a^2}\left(1 + c^2 \frac{k_T^2}{\omega_p^2}\right)} \tag{3.15}$$

$\overleftrightarrow{A}(\underset{\sim}{r}-\underset{\sim}{r}',\omega)$ is then obtained from $A(k,\omega)$ by means of the Fourier integral theorem

$$\overleftrightarrow{A}(\underset{\sim}{R},\omega) = \int \overleftrightarrow{A}(\underset{\sim}{k},\omega)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}} d^3\underset{\sim}{k}/(2\pi)^3 \tag{3.16}$$

where $\underset{\sim}{R} = \underset{\sim}{r}-\underset{\sim}{r}'$.

Note that in the expressions for A_{ij}, there appear five different denominators, four of which have singularities which <u>might</u> contribute to radiation.

We shall be interested in integrating these five expressions over all k-space. We need concern ourselves only with the denominators because the k_x, k_y and k_z appearing in the numerators may be taken outside of the integrals as appropriate partial derivatives since

k_x transforms to $-i\,\frac{\partial}{\partial x}$, etc.

Let us obtain the Fourier transforms of these five denominators which we shall call G_1 to G_5 inclusively. It should be noted that in later calculations, only those terms that fall off as 1/r shall be retained, as the main interest is in the radiation parts of the fields.

Thus, the transformed matrix elements take the form

$$A_{xx} = G_1 - \frac{c^2}{\omega_p^2} \frac{\partial^2}{\partial x^2} G_3 + (1 + \frac{1}{\gamma}) \frac{\partial^2}{\partial x^2} G_5 \qquad (3.17)$$

$$A_{yy} = G_1 - \frac{c^2}{\omega_p^2} \frac{\partial^2}{\partial y^2} G_3 + (1 + \frac{1}{\gamma}) \frac{\partial^2}{\partial y^2} G_5 \qquad (3.18)$$

$$A_{zz} = G_2 - \frac{c^2}{\omega_p^2} \frac{\partial^2}{\partial z^2} G_3 - (1 + \frac{1}{\gamma}) \frac{\partial^2}{\partial z^2} G_4 \qquad (3.19)$$

$$A_{xy} = A_{yx} = - \frac{c^2}{\omega_p^2} \frac{\partial^2}{\partial x \partial y} G_3 + (1 + \frac{1}{\gamma}) \frac{\partial^2}{\partial x \partial y} G_5 \qquad (3.20)$$

$$A_{xz} = A_{zx} = - \frac{c^2}{\omega_p^2} \frac{\partial^2}{\partial x \partial z} G_3 \qquad (3.21)$$

$$A_{yz} = A_{zy} = - \frac{c^2}{\omega_p^2} \frac{\partial^2}{\partial y \partial z} G_3 \qquad (3.22)$$

where

$$G_1 = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot R}}{k^2 - \omega^2/a^2} \qquad (3.23)$$

$$G_2 = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot R}}{k^2 + \omega_p^2/c^2} \qquad (3.24)$$

$$G_3 = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot R}}{k_z^2 - \frac{\omega^2}{a^2} - \frac{k_T^2}{\gamma}} \tag{3.25}$$

$$G_4 = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot R}}{(k^2 - \frac{\omega^2}{a^2})(k_z^2 - \frac{\omega^2}{a^2} - \frac{k_T^2}{\gamma})} \tag{3.26}$$

$$G_5 = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot R}}{(k^2 + \frac{\omega_p^2}{c^2})(k_z^2 - \frac{\omega^2}{a^2} - \frac{k_T^2}{\gamma})} \tag{3.27}$$

where

$$\gamma = \frac{\omega_p^2}{\omega^2} \frac{a^2}{c^2} . \tag{3.28}$$

We must specify the manner of evaluating the singularities in the inverse transforms G_1, G_3, G_4 and G_5 in order to have them correspond to real physical problems. The method is simple. An ambiguity in the inverse transform originates from the use of a Fourier integral not absolutely convergent. One must supplement the original integrand with a damping factor. For example,

$$G(\omega) = \int_{-\infty}^{\infty} dT\, e^{i\omega T}\, G(T)\, e^{-\varepsilon |T|} \qquad \varepsilon > 0 \tag{3.29}$$

where $T = t-t'$ is the usual choice. In addition, we know that $G(T)$ relates the field at time t to a source at time t'. Causality implies that $G(t-t') = 0$ if $t < t'$ $(T < 0)$. Hence,

$$G(\omega) = \int_0^{\infty} dT\, e^{i\omega T}\, e^{-\varepsilon T}\, G(T)$$

$$= \int_0^{\infty} dT\, e^{i(\omega+i\varepsilon)T}\, G(T) \quad . \qquad (3.30)$$

Thus, for causal functions, the necessary modification is to replace

$$G(\omega) \longrightarrow G(\omega+i\varepsilon) \quad . \qquad (3.31)$$

Then the integral (3.30) is absolutely convergent, and in the final calculation, one can let $\varepsilon \to +0$ after it has been used to define the singularities.

<u>Evaluation of G_1</u>

$$G_1(\underset{\sim}{R},\omega) = \int \frac{e^{ik\cdot \underset{\sim}{R}}}{k^2 - \omega^2/a^2} \frac{(d^3k)}{(2\pi)^3} \quad . \qquad (3.32)$$

This is the well-known radiation Green's function propagator[4]

$$G_1 = \frac{e^{i(\omega/a)R}}{4\pi R} \quad . \qquad (3.33)$$

where R is the magnitude of $\underset{\sim}{R}$.

Since the solution of G_1 is known, it is advantageous to put the integral in Equation (3.32) into another form which will be encountered repeatedly in future calculations.

The denominator of the integrand in (3.32) can be "exponentiated." If as discussed above, the ω in the denominator is replaced by $(\omega+i\varepsilon)$,

$$G_1 = \int \frac{(d^3k)}{(2\pi)^3} \int_0^{\infty} i ds\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}} \exp\left[-is\left(k^2 - \left(\frac{\omega^2-\varepsilon^2}{a^2}\right)\right)\right] \exp\left(-2\,\frac{\varepsilon\omega}{a^2}\, s\right) \qquad (3.34)$$

The $e^{-\frac{2\varepsilon\omega s}{a^2}}$ serves as a damping factor and makes the integral (3.34) absolutely convergent.

Letting $\varepsilon \to 0$, all terms involving ε can be disregarded and

$$G_1 = \int \frac{d^3\underset{\sim}{k}}{(2\pi)^3} \int_0^\infty ids\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}} \exp -i(k^2 - \frac{\omega^2}{a^2})s \quad . \qquad (3.35)$$

The k-integral is evaluated by completing the square in the exponential

$$G_1 = \int_0^\infty \frac{ids}{(2\pi)^3} \exp(is\frac{\omega^2}{a^2}) \exp(i\frac{R^2}{4s}) \int \exp\left[-is(k + \frac{R}{2s})^2\right] d^3\underset{\sim}{k}$$

$$= \frac{i^{-1/2}}{8\pi^{3/2}} \int_0^\infty ds\, s^{-3/2} \exp(i\frac{\omega^2}{a^2}s) \exp(i\frac{R^2}{4s}) \qquad (3.36)$$

However, since G_1 is known, the solution for this particular type of integral in (3.36) can be obtained.

In general for $A > 0$, $B > 0$.

$$\int_0^\infty ds\, s^{-3/2} e^{isA^2} e^{i\frac{B^2}{4s}} = \frac{\sqrt{\pi i}\, e^{iAB}}{\frac{B}{2}} \quad . \qquad (3.37)$$

This result, Equation (3.37), will be used in subsequent calculations.

Evaluation of G_2

The second Green's function G_2 is recognized as the Fourier transform of the Debye-Yukawa potential

$$G_2 = \frac{\exp(-\frac{\omega_p}{c} R)}{4\pi R} \tag{3.38}$$

The denominator is "exponentiated" to put the integral into a form to be compared with later results. In this case, as contrasted with G_1, the denominator is not singular and thus a real exponential can be used.

$$G_2 = \int_0^\infty dt \int \frac{(d^3k)}{(2\pi)^3} e^{ik\cdot R} e^{-k^2 t} \exp(-\frac{\omega_p^2}{c^2} t) \tag{3.39}$$

Again, the k-integral can be evaluated by completing the square in the exponential. Hence

$$G_2 = \frac{1}{8\pi^{3/2}} \int_0^\infty \exp(-\frac{\omega_p^2}{c^2} t) \exp(-\frac{R^2}{4t}) t^{-3/2} dt \tag{3.40}$$

Since G_2 is known, a solution for the integral in (3.40) is easily obtained. For $C > 0$, $D > 0$

$$\int_0^\infty e^{-C^2 t} e^{-\frac{D^2}{4t}} t^{-3/2} dt = \frac{\sqrt{\pi}\, e^{-CD}}{\frac{D}{2}} . \tag{3.41}$$

This result, Equation (3.41), will also be used later.

Evaluation of G_3

The determination of the inverse transform

$$G_3 = \int \frac{(d^3k)}{(2\pi)^3} \frac{e^{ik\cdot R}}{k_z^2 - \frac{\omega^2}{a^2} - \frac{k_T^2}{\gamma}} \tag{3.42}$$

proceeds along lines similar to those of G_1 and G_2. It shall be shown shortly that two cases must be considered, one of which is analogous to the radiation Green's function G_1, the other related to the Debye-Yukawa potential, G_2.

Let k_T denote that portion of the propagation vector, k, which is transverse to the magnetic field direction. Similarly, let R_T denote that part of the vector R which is perpendicular to the magnetic field direction.

"Exponentiating" the denominator in (3.43) and arranging terms, one obtains

$$G_3 = \int_0^\infty \frac{ids}{(2\pi)^3} \int dk_z\, e^{ik_z Z}\, e^{ik_z^2 s} \int d^2k_T\, e^{ik_T R_T} \exp(i\frac{k_T^2}{\gamma} s) \exp(i\frac{\omega^2}{a^2} s) \; . \tag{3.43}$$

Completing the square in both the k_T and k_z integrals leads to

$$G_3 = \int_0^\infty \frac{ids}{(2\pi)^3} \left(\frac{\pi}{is}\right)^{1/2} \frac{\pi\gamma}{-is} \exp(i\, \frac{\omega^2}{a^2}\, s) \exp\left[\frac{i}{4s}(Z^2 - R_T^2\gamma)\right] \tag{3.44}$$

Let $Z^2 - R_T^2\gamma = \bar{R}^2$.

If $\bar{R}$ is positive, the integrand in Equation (3.44) is exactly the same as the integrand in Equation (3.37), with $A^2 = \omega^2/a^2$ and $B^2 = \bar{R}^2$.

Since the solution of the integral in Equation (3.37) is known, one can immediately write down the solution of (3.44)

$$G_3 = \frac{-\gamma\, e^{i\frac{\omega}{a}\bar{R}}}{4\pi\bar{R}} = -\frac{\gamma \exp(i\, \frac{\omega}{a} \sqrt{Z^2 - R_T^2\gamma})}{4\pi\sqrt{Z^2 - R_T^2\gamma}} \; . \tag{3.46}$$

This is analogous to the radiation Green's function G_1, except that the isotropic radius vector R has been replaced by the quantity $\bar{R}$. If $\bar{R}^2$ is negative -- consider the s-integral as a contour integral along the positive real axis in the complex plane.

If one integrates along the positive real axis from 0 to Q (see Figure 1), and then along a circle of radius r_o in the first quadrant of the complex plane to P and n takes the limit as $r_o \rightarrow \infty$, no additional contribution is made to the original s-integral.

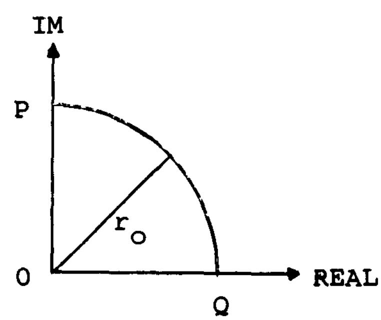

Figure 1

For along the curved path OP, s is replaced in the integrand by $r_o e^{i\theta}$ where $0 \leq \theta \leq \frac{\pi}{2}$.

The term $e^{i(\omega^2/a^2)s}$ now contains a damping factor for all θ as r_o ∞.

The term

$$e^{i\bar{R}^2/4s} = \exp \frac{i\bar{R}^2}{4r} e^{-i\theta} \tag{3.47}$$

goes to unity for all values of θ, as $r_o \rightarrow \infty$. Hence, the extra integral gives no contribution. Note that there are no singularities <u>within</u> the quadrant to prevent use from deforming the path of integration to the positive imaginary axis.

Hence, if $\bar{R}^2 < 0$, i.e., $R_{T\gamma}^2 > Z^2$, there is no contribution to the s-integral along OP.

If, instead of integrating along OQ and then QP the integration is performed directly along the imaginary axis from O to P, s is replaced by it in the integrand in Equation (3.44),

$$G_3 = \frac{\gamma}{8\pi^{3/2}} \int_0^{\infty} \exp(-\frac{\omega^2}{a^2} t) \exp(+ \frac{\bar{R}^2}{4t}) t^{-3/2} \quad . \tag{3.48}$$

But, for $\bar{R}^2 < 0$, this integral is exactly the same as the integral expression of the Debye-Yukawa potential as given by Equation (3.41) with $C^2 = \omega^2/a^2$ and $D^2 = -\bar{R}^2$. Therefore

$$G_3 = \frac{\gamma e^{-\frac{\omega}{a}\sqrt{\gamma R_T^2 - z^2}}}{4\pi\sqrt{\gamma R_T^2 - z^2}} \quad . \tag{3.49}$$

What is the physical significance of the conditions $\bar{R}^2 > 0$ or $\bar{R}^2 < 0$?

$\bar{R}^2 = 0$, i.e., $z^2 = R_T^2\gamma$ defines a cone in a three-dimensional space. If for simplicity it is assumed that the source is located at the origin ($r_T' = 0$; $z' = 0$), then this cone is further defined by

$$z^2 = r_T^2\gamma \tag{3.50}$$

where $r^2 = z^2 + r_T^2$ is the distance to the field point. Inside the cone, where $z^2 > r_T^2\gamma$, the fields as given by G_3 are radiation-like expressions which fall off as $(z^2-r_T^2\gamma)^{-1/2}$, similar to the 1/r dependence of a typical radiation field. On the surface of the cone the field is singular and infinite. Outside of the cone, where $r_T^2\gamma > z^2$, the fields, as given by G_3, are damped exponentially and do not correspond to radiation. Therefore, if there is any radiation associated with the Green's function G_3, it is contained within the cone defined by $z^2 = r_T^2\gamma$.

Evaluation of G_4

The denominator in the integral expression for G_4 is a product of the terms appearing in the expressions for G_1 and G_3 respectively.

$$G_4 = \int \frac{(d^3\underset{\sim}{k})}{(2\pi)^3} \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}}}{k^2 - \frac{\omega^2}{a^2}} \frac{1}{k_z^2 - \omega^2/a^2 - \frac{k_T^2}{\gamma}} \tag{3.51}$$

Each term in the denominator can be "exponentiated" and when like terms are grouped

$$= -\frac{1}{(2\pi)^3} \int_0^\infty dt \int_0^\infty ds \int dk_z \; \exp(ik_z Z) \exp(-ik_z^2(s+t))$$

$$\cdot \int d^2k_T \; \exp(ik_T R_T) \exp\left[-ik_T^2\left(s - \frac{t}{\gamma}\right)\right] \exp\left[i\frac{\omega^2}{a^2}(s+t)\right] \tag{3.52}$$

Again, the k_z and k_T integrals can be evaluated by completing the squares in the exponents

$$G_4 = \frac{-i^{-3/2}}{8\pi^{3/2}} \int_0^\infty dt \int_0^\infty ds \frac{\exp\left[\frac{i\,R_T^2}{4(s-\frac{t}{\gamma})}\right] \exp\left[\frac{i\,Z^2}{4(s+t)}\right] \exp\left[i\,\frac{\omega^2}{a^2}(s+t)\right]}{(s-\frac{t}{\gamma})\,(s+t)^{1/2}} \tag{3.53}$$

Now consider a double change of variable.

Let

$$s+t = u \tag{3.54}$$

$$s - \frac{t}{\gamma} = v \quad . \tag{3.55}$$

The Jacobian of this tranformation is

$$J(s,t,u,v) = \frac{\gamma}{1+\gamma} \quad . \tag{3.56}$$

In determining the new limits of integration, it is noted that the infinite quarter plane, bounded by the positive s and t axes, must be covered by integrating along a pair of oblique axes, u and v. This is accomplished by integrating over v from the line s = 0, which corresponds to $v = -u/\gamma$, to the line t = 0 which corresponds to v = u, and then along the u-axis from zero to infinity

$$G_4 = - \frac{i^{-3/2}}{8\pi^{3/2}} \int_0^\infty du\ u^{-1/2} \exp(i\,\frac{\omega^2}{a^2}\,u) \exp(i\,\frac{z^2}{4u}) \int_{-u/\gamma}^{u} dv\ \frac{\exp(i\,\frac{R_T^2}{4\gamma})}{v}\ \frac{\gamma}{1+\gamma} \tag{3.57}$$

Consider the special case of $R_T \rightarrow 0$. This corresponds to the source being located at the origin and an observer looking along the magnetic field direction. In the v-integral, for finite R_T the singularity at v = 0 is damped out symmetrically for $|v| \leq R_T^2/4$ by the presence of the exponential. This defines the Cauchy principle value of the integral. In the limiting case of $R_T \rightarrow 0$, in order for the integral to converge, one must bear in mind that we are concerned with its principal value

$$P \int_{-u/\gamma}^{u} \frac{dv}{v} = \ell n\ |v| \Big|_{-u/\gamma}^{u} = \ell n\ \gamma \quad . \tag{3.58}$$

Then

$$G_4(R_T = 0) = - \frac{\gamma}{1+\gamma}\,\ell n\,\gamma\,\frac{i^{-3/2}}{8\pi^{3/2}} \int_0^\infty du\ u^{-1/2} \exp(i\,\frac{\omega^2}{a^2}\,u) \exp(i\,\frac{z^2}{4u}) \tag{3.59}$$

Making the substitution u = 1/x, the remaining integral takes the form

$$\int_0^\infty dx\, x^{-3/2} \exp(i\, \frac{Z^2}{4}\, x) \exp(i\, \frac{\omega^2}{a^2 x}) \quad .$$

This integral has been encountered previously in the calculation of G_1 and G_3 and its solution is given by Equation (3.37).

For the present case

$$A^2 = \frac{Z^2}{4} \qquad\qquad B^2 = \frac{4\omega^2}{a^2} \quad . \tag{3.60}$$

Therefore, from Equation (3.37)

$$\int_0^\infty dx\, x^{-3/2} \exp(i\, \frac{Z^2}{4}\, x) \exp(i\, \frac{\omega^2}{a^2 x}) = \frac{\sqrt{\pi i}}{\omega/a} \exp(i\, \frac{\omega}{a}\, Z) \tag{3.61}$$

Then

$$G_4(R_T = 0) = \frac{i\gamma}{1+\gamma}\, \ln \gamma\, \frac{\exp(i\, \frac{\omega}{a}\, Z)}{8\pi\, \frac{\omega}{a}} \tag{3.62}$$

This expression differs from the previous radiation-like Green's functions that have been calculated. $G_4(R_T = 0)$ represents a plane wave propagating along the magnetic field axis. There is no decrease in intensity with distance away from the origin. This result is not completely unexpected. G_1 represents an isotropic mode of radiation and is analogous to the isotropic mode derived in Chapter I for the case of zero sound velocity (Equation (1.13)). G_3, for $\bar{R}^2 > 0$, represents radiation filling a cone of small angle about the magnetic field line and is related to the one-dimensional Alfven mode of Chapter I (1.14) which is an idealization of the actual physical situation. G_4 is a measure of the radiation resulting from the interaction of these two aforementioned modes.

For the case of $R_T \neq 0$, an interesting integral representation of G_4 can be obtained.

If the partial derivatives of G_4 is taken with respect to R_T^2, Equation (3.56) becomes

$$\frac{\partial}{\partial R_T^2} G_4 = \frac{1}{32\pi^{3/2}} \int_0^\infty du\, u^{-1/2}\, e^{\frac{i\omega^2}{a^2} u}\, e^{\frac{iZ^2}{4u}} \int_{-u/\gamma}^{u} dv\, \frac{e^{i\frac{R_T^2}{4v}}}{v^2} \frac{\gamma}{1+\gamma}$$

(3.63)

The integral over v can now be performed since the differential of the exponent appears in the integrand.

$$\frac{\partial^2}{\partial R_T^2} G_4 = \frac{i^{-1/2}}{32\pi^{3/2}} \int_0^\infty du\, u^{-1/2} \exp(i\frac{\omega^2}{a^2}) \exp(i\frac{Z^2}{4u}) \left[-\frac{4}{iR_T^2} \exp(i\frac{R_T^2}{4v}) \right]_{-u\gamma}^{u} \frac{\gamma}{1+\gamma}$$

$$= \frac{i^{-3/2}}{8\pi^{3/2}} \frac{\gamma}{1+\gamma} \int_0^\infty \frac{du}{R_T^2}\, u^{-1/2} \exp(i\frac{\omega^2}{a^2}) \exp(i\frac{Z^2}{4u}) \left[\exp(i\frac{R_T^2}{4u}) - \exp(-i\frac{R_T^2\gamma}{4u}) \right]$$

(3.64)

If again the substition u = 1/x is made, both expressions in the integrand can be put into a form similar to that of the integral in Equation (3.37), our known identity.

$$= \frac{i^{-3/2}}{8\pi^{3/2} R_T^2} \frac{\gamma}{1+\gamma} \int_0^\infty dx\, x^{-3/2} \left[\exp(i\frac{\omega^2}{a^2 x}) \exp(\frac{i}{4}\sqrt{Z^2+R_T^2 x}) - \exp(\frac{i}{4}\sqrt{Z^2-R_T^2\gamma x}) \right]$$

(3.65)

$$= \frac{-i}{8\pi R_T^2 \frac{\omega}{a}} \frac{\gamma}{1+\gamma} \left[\exp(i \frac{\omega}{a} \sqrt{Z^2 + R_T^2}) - \exp(i \frac{\omega}{a} \sqrt{Z^2 - R_T^2\gamma})\right] \quad (3.66)$$

The Green's function is then obtained by integrating over all values of R_T^2.

$$G_4 = \int_0^{R_T^2} d(R_T^2) \left(-\frac{\gamma}{1+\gamma}\right) \frac{1}{8\pi \frac{\omega}{a}} \frac{1}{R_T^2} \left[\exp(i \frac{\omega}{a} \sqrt{Z^2 + R_T^2}) - \exp(i \frac{\omega}{a} \sqrt{Z^2 - R_T^2\gamma})\right]$$

$$+ \frac{\exp(i \frac{\omega}{a} Z}{8\pi \frac{\omega}{a}} \frac{i\gamma}{1+\gamma} \cdot \ell n \ \gamma \quad (3.67)$$

where the term outside of the integral is the value of the Green's function for $R_T = 0$, which has been previously calculated (Equation (3.62)

It is seen from this integral expression for G_4, that for $R_T^2\gamma < Z^2$, the Green's function is radiation-like, and that it is partially the sum of contributions over cones described by $Z^2 = R_T^2\gamma$ (for fixed Z). There is a contribution at $R_T = 0$ which represents a plane wave solution on the magnetic field axis.

Evaluation of G_5

The denominator in the integral expression for G_5 is a product of the terms appearing in the expressions for G_2 and G_3. Physically, this Green's function represents the interaction of the directional mode represented by G_3 inside of the cone $Z^2 = R_R^2\gamma$ with the attenuated mode represented by the Debye-Yukawa potential. One expects the product of the two to be a directional damped oscillation

$$G_5 = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot R}}{k_z^2 + \frac{\omega_p^2}{c^2} k_z^2 - \omega^2/a^2 - \frac{k_T^2}{\gamma}} \quad \cdot \quad (3.68)$$

Each term in the denominator can be "exponentiated." An imaginary exponential must be used for the term containing the singularity.

$$G_5 = \frac{i}{(2\pi)^3} \int_0^\infty ds \int_0^\infty dt \int dk_z \exp\left[-k_z^2(s+it)\right] e^{ik_z Z}$$

$$\cdot \int d^2\, k_T\, e^{ik_T R_T} \exp\left[-k_T^2(s-i\frac{t}{\gamma})\right] \exp(-\frac{\omega_p^2}{c^2}s) \exp(i\frac{\omega^2}{a^2}t) \qquad (3.69)$$

$$\frac{i}{8\pi^{3/2}} \int_0^\infty ds \int_0^\infty dt \frac{\exp\left[-\frac{\omega_p^2}{c^2}(s-i\frac{t}{\gamma})\right] \exp\left[-\frac{Z^2}{4(s+it)}\right] \exp\left[-\frac{R_T^2}{4(s-i\frac{t}{\gamma})}\right]}{(s-i\frac{t}{\gamma})(s+it)^{1/2}} \qquad (3.70)$$

Previously, we have obtained G_4 exactly for the case of $R_T = 0$. In the present example, we shall consider the case when $Z = 0$ and show that $G_5(Z = 0)$ can be determined in an analogous manner.

Both s and t are real variables. If the upper limit of the t-integration is extended to cover the circle in the first quadrant by $\lim_{r_o \to \infty} r_o\, e^{i\theta}$, it is easily seen that there is no contribution to the t-integral along the extra portion of the path. For, if $t \to t + i\tau$, $\tau > 0$, the term $\exp i\, \omega_p^2/c^2\, t$ contributes $e^{-\omega_p^2/c^2\, \tau}$ which approaches zero as $\tau \to \infty$, and the term $\exp(R_T^2/4(s-i\frac{t}{\gamma}))$ is bounded in this limit. Then, as was shown previously in the calculation of G_3, the contour can be deformed and the integration performed along the positive imaginary axis.

It should be noted that the contour cannot be deformed in this manner when $Z \neq 0$.

For then, the term $e^{\frac{iZ^2}{4(s+it)}}$ becomes an isolated essential singularity at a point on the path of integration -- namely where $t = is$, and gives rise to an infinite contribution to the integral.

For the Z = 0 case, this term vanishes and the above difficulty is circumvented.

Then t can be replaced in the integrand of Equation (3.70) by $i\tau$.

$$G_5 = -\frac{1}{8\pi^{3/2}} \int_0^\infty ds \int_0^\infty d\tau \frac{\exp\left[-\frac{\omega_p^2}{c^2}\left(s+\frac{\tau}{\gamma}\right)\right] \exp\left[-\frac{Z^2}{4(s-\tau)}\right] \exp\left[-\frac{R_T^2}{4\left(s+\frac{\tau}{\gamma}\right)}\right]}{\left(s+\frac{\tau}{\gamma}\right)(S-\tau)^{1/2}} . \quad (3.71)$$

One now proceeds along lines similar to those used in the evaluation of G_4. Introduce the double change of variable

$$s-\tau = u \quad (3.72)$$

$$s+\frac{\tau}{\gamma} = v \quad . \quad (3.73)$$

The Jacobian of the transformation is again $\frac{\gamma}{1+\gamma}$.

The new limits of integration are:

Along u from $-\gamma v$ to v

Along v from 0 to ∞.

Thus

$$G_5 = -\frac{\gamma}{\gamma+1} \frac{1}{8\pi^{3/2}} \int_0^\infty dv \frac{\exp\left[-\frac{\omega_p^2}{c^2} v\right] \exp\left[-\frac{R_T^2}{4v}\right]}{v} \int_{-\gamma v}^{v} \frac{du}{u^{1/2}} . \quad (3.74)$$

The u-integral has an integrable singularity and can be performed directly

$$\int_{-\gamma v}^{v} \frac{du}{u^{1/2}} = 2v^{1/2}\left[1 - i\sqrt{\gamma}\right]$$

$$G_5 = \frac{1}{4\pi^{3/2}} \frac{\gamma}{1+i\sqrt{\gamma}} \int_0^\infty dv\; v^{-1/2} \exp\left(-\frac{\omega_p^2}{c^2} v\right) \exp\left(-\frac{R_T^2}{4v}\right) . \tag{3.75}$$

If one now makes the substitution $v = \frac{1}{x}$, the resulting integrand is exactly the same as that in Equation (3.41), the integral expression for the Debye-Yukawa potential. Therefore

$$G_5(Z=0) = \frac{-1}{4\pi^{3/2}} \frac{\gamma}{1+i\sqrt{\gamma}} \int_0^\infty dx \frac{\exp\left(-\frac{\omega_p^2}{c^2 x}\right) \exp\left(-\frac{R_T^2}{4} x\right)}{x^{3/2}} \tag{3.76}$$

and from Equation (3.41)

$$= \frac{-1}{4\pi} \frac{\gamma}{1\mp i\sqrt{\gamma}} \frac{\exp\left(-\frac{\omega_p}{c} R_T\right)}{\frac{\omega_p}{c}} . \tag{3.77}$$

Thus, $G_5(Z=0)$ represents an "attenuated plane wave" traveling in the direction transverse to the magnetic field lines. This result is not unexpected, since it represents the "interference" between the damped Debye-Yukawa potential and the exponentially attenuated oscillation given by G_3 outside of the cone $Z^2 = R_T^2\gamma$.

As discussed previously, the deformation of the contour is not valid for $Z \neq 0$, and one cannot proceed in the manner described above. Therefore, in order to get an expression for $G_5(Z \neq 0)$, consider the following scheme

$$G_5 = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}}}{(k^2 + \frac{\omega_p^2}{c^2})(k_z^2 - \frac{\omega^2}{a^2} - \frac{k_T^2}{\gamma})} \tag{3.78}$$

$$k_z^2 - \frac{\omega^2}{a^2} - \frac{k_T^2}{\gamma} = \frac{-1}{\gamma}\,(k^2 + \frac{\omega_p^2}{c^2} - (\gamma+1)k_z^2) \quad . \tag{3.79}$$

Therefore,

$$G_5 = -\gamma \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}}}{(k^2 + \frac{\omega_p^2}{c^2})\left[k^2 + \frac{\omega_p^2}{c^2} - (\gamma+1)k_z^2\right]} \quad . \tag{3.80}$$

Now decompose the product in the denominator into a sum of terms, by the method of partial fractions

$$G_5 = \frac{\gamma}{\gamma+1} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}}}{k_z^2} \left[\frac{1}{k^2 + \frac{\omega_p^2}{c^2}} - \frac{1}{k^2 + \frac{\omega_p^2}{c^2} - (\gamma+1)k_z^2}\right] \tag{3.81}$$

and making use of the Fourier transform relationship between k_z and $\frac{\partial}{\partial z}$

$$-\frac{\partial^2}{\partial z^2} G_5 = \frac{\gamma}{\gamma+1} \int \frac{d^3k}{(2\pi)^3} e^{i\underset{\sim}{k}\cdot\underset{\sim}{R}} \left(\frac{1}{k^2 + \frac{\omega_p^2}{c^2}} - \frac{1}{k^2 + \frac{\omega_p^2}{c^2} - (\gamma+1)k_z^2}\right)$$

$$= \frac{1}{\gamma+1}\left[G_2 + \frac{1}{\gamma} G_3\right] \tag{3.82}$$

from the integral expressions for the transforms of G_2 and G_3 . Therefore,

$$- \frac{\partial^2}{\partial Z^2} G_5 = \frac{\gamma}{1+\gamma} \left[\frac{\exp(-\frac{\omega_p}{c} R)}{4\pi R} - \frac{\exp(i \frac{\omega}{a} \bar{R})}{4\pi \bar{R}} \right] = \qquad (3.83)$$

The next task is to find the Green's function for the operator $-\frac{\partial^2}{\partial Z^2}$; $g(Z,Z')$, where

$$\frac{\partial^2}{\partial Z^2} g(Z,Z') = \delta(Z-Z') \quad . \qquad (3.84)$$

Then

$$G_5 = - \int g(Z,Z') f(X,Y,Z') dZ' \quad . \qquad (3.85)$$

From the physics of the problem, we know that

$$g(Z,Z') = g(Z-Z') \qquad (3.86)$$

$g(Z-Z')$ must satisfy $\partial^2/\partial Z^2 \, g(Z-Z') = 0$ everywhere except at $Z = Z'$ and the change in the first derivative must be +1 as one passes through the point $Z = Z'$ from above. This Green's function is easily obtained. Consider the function

$$\frac{1}{2} \left| Z - Z' \right| \quad . \qquad (3.87)$$

Its slope changes by +1 as the point $Z = Z'$ is traversed from above, and its second derivative is zero everywhere for $Z \neq Z'$. Therefore, this function satisfies $\partial^2/\partial Z^2 \left| Z - Z' \right| = \delta(Z - Z')$. We shall see that

$$g(Z - Z') = \frac{1}{2} \left| Z - Z' \right| \qquad (3.88)$$

without any further homogeneous contribution gives the previously calculated Z = 0 result. Then

$$G_5(Z) = -\frac{1}{2}\frac{\gamma}{\gamma+1}\int_{-\infty}^{Z}(Z - Z')\left[G_2(Z') + \frac{1}{\gamma}G_3(Z')\right]dZ'$$

$$-\frac{1}{2}\frac{\gamma}{\gamma+1}\int_{Z}^{\infty}(Z' - Z)\left[G_2(Z') + \frac{1}{\gamma}G_3(Z')\right]dZ' \; .$$

(3.89)

G_3, it must be remembered, has two functional forms, one denoting propagation and the other damping. The appropriate form of G_3 must be used in the integrals in Equation (3.89), depending on whether the value of $|Z|$ in the limits is greater than or less than $R_T\sqrt{\gamma}$. Therefore, Equation (3.89) can be explicitly written

$$G_5 = \frac{\gamma}{\gamma+1}\frac{1}{4\pi}\left[\frac{\exp\left(i\frac{\omega}{a}\sqrt{Z^2 - R_T^2\gamma}\right)}{i\frac{\omega}{a}} - \frac{\exp\left(-\frac{\omega_p}{c}\sqrt{Z^2 + R_T^2}\right)}{\frac{\omega_p}{c}}\right]$$

$$+\frac{Z}{4\pi}\frac{\gamma}{\gamma+1}\int_{0}^{\infty}dZ'\left[\frac{\exp\left(-\frac{\omega_p}{c}\sqrt{Z'^2 + R_T^2}\right)}{\sqrt{Z'^2 + R_T^2}} + \frac{\exp\left(i\frac{\omega}{a}\sqrt{Z'^2 - R_T^2\gamma}\right)}{\sqrt{Z'^2 - R_T^2\gamma}}\right]$$

for $|Z| > R_T\sqrt{\gamma}$ (3.90)

and

$$G_5 = -\frac{\gamma}{\gamma+1}\frac{1}{4\pi}\left[\frac{\exp\left(-\frac{\omega_p}{c}\sqrt{R_T^2+Z^2}\right)}{\frac{\omega_p}{c}} + \frac{\exp\left(-\frac{\omega}{a}\sqrt{R_T^2\gamma - Z^2}\right)}{\frac{\omega}{a}}\right]$$

$$+\frac{Z}{4\pi}\frac{\gamma}{\gamma+1}\int_0^\infty dZ'\left[\frac{\exp\left(-\frac{\omega_p}{c}\sqrt{R_T^2+Z'^2}\right)}{\sqrt{R_T^2+Z'^2}} + \frac{\exp\left(-\frac{\omega}{a}\sqrt{R_T^2\gamma - Z'^2}\right)}{\sqrt{R_T^2\gamma - Z'^2}}\right]$$

$$\text{for } |Z| < R_T\sqrt{\gamma} \; . \qquad (3.91)$$

In writing down Equations (3.90) and (3.91) we have made use of the boundary condition that all the exponentials in the integrand approach zero at $Z' = \pm\infty$. This corresponds to the conditions of the actual physical problem where even the radiation modes are not infinite but go to zero at a sufficiently large distance from the source. It is seen from Equation (3.90) that the first term in that expression does represent radiation. Thus, the interaction of the radiation term

$$-i\gamma\frac{e^{i\frac{\omega}{a}\sqrt{Z^2-R_T^2\gamma}}}{4\pi\sqrt{Z^2-R_T^2\gamma}}$$

with the damped mode described by the Yukawa potential, leads to a radiation-like term inside the cone $Z^2 = R_T^2\gamma$, in G_5.

The reader can easily verify that Equation (3.91) reduces to the result of our previous calculation for $Z = 0$, [c.f. Eq. (3.83)],

$$G_5(Z=0) = \frac{\gamma}{1+i\sqrt{\gamma}}\frac{1}{4\pi}\frac{e^{-\frac{\omega_p}{c}R_T}}{\frac{\omega_p}{c}} \; .$$

The radiation problem has been so set up, that now with the elements of the matrix A determined, one must merely evaluate a single current source $j(r,t)$. In Chapter VI, the radiation fields arising from several different types of current excitations will be calculated and discussed regarding their physical significance.

CHAPTER IV

RADIATION FROM UNIFORMLY MOVING CHARGES

Among the processes by which charged particles are slowed down in passing through solid and liquid material media, Cerenkov radiation is one of the most efficient. Such a state of affairs might be true also for charged particles passing through ionized magnetized 'fluids', or plasmas, for which the conditions resemble, in some respects, the conditions needed for Cerenkov radiation. It will be the purpose of the next two chapters to reveal the analogy in more detail, and to estimate the quantitative importance of the results. This work bears on the dynamics of the ionized zones surrounding the earth, but is too restricted to be applied directly to that problem. The charged particles which originate on the sun, and reach the earth's magnetosphere, arrive in bursts of plasma of densities perhaps comparable to that of the ambient plasma, rather than as independent particles. What such <u>streams</u> do upon reaching the vicinity of the earth must be investigated by other means.

A moving charged particle does not radiate in traversing empty space unless it is accelerated. The latter possibility is realized if the particle has a component of velocity across the lines of the magnetic field. But such "cyclotron radiation" is usually important only for extreme relativistic motions, because the accelerations involved are not very great. The most efficient cases of the radiation resulting from single accelerated particles occur when fast particles are brought to rest by dense solids ("bremsstrahlung"): the fast particle is deflected strongly by the Coulomb fields of the heavy nuclei, and radiates considerably in virtue of this acceleration. But for gases, and plasmas, this effect is also to be considered an unimportant mechanism for the "stopping" of the particle.

One must turn to "collective behavior" to find an important energy-loss mechanism. In crossing the plasma, the charged particle accelerates the ions in its neighborhood. While no one of the individual ions is accelerated greatly, there are, however, many which are set into motion, and with definite phase relations, so that the actual fields can be quite considerable. This radiation of the induced currents set up in a solid transparent substance was first observed (accidentally) by Cerenkov in 1934. Its existence had not been suspected even though closely related calculations had previously been made of the field around a charge moving faster than the velocity of light.[10] The field lies entirely in a cone behind the particle, and is singular on the surface of the cone. The Poynting vector at the surface is normal to the cone and outwards, but the rate of radiation could not be calculated. In any case, the theory of relativity (that followed the calculation) eliminated the possibility of charges moving faster than light. Franck and Tamm in 1937 explained Cerenkov's discovery by treating a charge moving faster than the velocity of light <u>in the medium</u>. All of the details of the medium were compressed into one function -- the frequency dependent phase velocity of light (or the index of refraction). Their treatment of the problem is, therefore, often referred to as the "macroscopic approach," since the only information needed about the material is its index of refraction for light of different frequencies, a bulk property of the medium. It is a method of extreme generality, however, being restricted primarily to a linear approximation to the response of the medium to excitations. But within the domain of linearity, one can treat nearly all important problems of the interaction of charged particles and radiation with neutral and ionized substances.[11]

As is well known from the work of Alfven,[12] an <u>incompressible</u> conducting fluid in a DC magnetic field has a one-dimensional, low-frequency mode of propagation of electromagnetic energy -- a transverse field propagating along the lines of force of the magnetic field at a velocity <u>a</u> which can be considerably less than the velocity of light in vacuum (<u>c</u>). This Alfven (hydromagnetic) mode is not suitable for Cerenkov radiation, which is a charateristic of an "isotropic" mode -- that is, one which propagates equally well in all directions. The reason for this is that the instantaneous Cerenkov radiation lies very nearly <u>in the surface of a cone of finite angle</u>. At the very best, the direction of the magnetic field lines (and therefore of the propagation vector of the Alfven wave) could be parallel to only one generating line of this cone; there can be no finite density of radiation in one line.

For a compressible fluid, however, there exists in addition to the Alfven mode an "isotropic" mode whose phase velocity is the same as the Alfven velocity. Since it depends on the presence of compressibility, it is often referred to as a "hydromagnetic sound wave."[13] It is <u>this</u> mode that will be investigated for Cerenkov-like radiation. It is well defined so long as the actual sound velocity in the plasma, <u>s</u>, is much less than <u>a</u>; a larger value of <u>s</u> corresponds to a more rigid, incompressible medium. For the zero temperature gas we have in mind, <u>s</u> is near zero. The difference between the ordinary Cerenkov effect and the hydromagnetic example lies in the interference of the one-dimensional mode with the "isotropic" mode.

ORDINARY CERENKOV RADIATION

We have discovered that the tensor $X(k\omega)$ characterizes the electrodynamical properties of a "slightly disturbed" medium. For most solids and liquids (in the frequency regions for which they are transparent) $X(k\omega)$ can be adequately represented by a positive, diagonal, isotropic tensor which depends only on ω:

$$X(k\omega) \longrightarrow X_o(\omega)\,1 \tag{4.1}$$

Consequently, the propagation denominator $k^2 - \frac{\omega^2}{c^2}$ becomes $k^2 - \frac{\omega^2}{c^2}(1 + X(\omega))$.

The quantity

$$a(\omega) = \sqrt{\frac{c^2}{1 + X_o(\omega)}} < c \tag{4.2}$$

is the "phase velocity" for light at frequency ω. In terms of the notation of Chapter II, we can show the following in this case:

$$M = \frac{1}{k^2 - \omega^2/a^2} \tag{4.3}$$

$$1 - k.M.k = \frac{-\omega^2/a^2}{k^2 - \omega^2/a^2} \tag{4.4}$$

$$A = \frac{1}{k^2 - \omega^2/a^2} \left(1 - \frac{a^2}{\omega^2} k\, k\right) \tag{4.5}$$

$$E = \frac{i\mu_o}{k^2 - \omega^2/a^2} \left(\omega j^{ext}(k\omega) - a^2 k\, \rho^{ext}(k\omega)\right) \tag{4.6}$$

$$B = \frac{1}{\omega} k \times E = \frac{i\mu_o}{k^2 - \omega^2/a^2} k \times j^{ext}(k\omega) \tag{4.7}$$

We have used the conservation law:

$$k \cdot j^{ext} = \omega\, \rho^{ext} \tag{4.8}$$

The radiation of the external sources may be found from the Poynting vector:

$$S(rt) = \frac{1}{\mu o} E(rt) \times B(rt) \tag{4.9}$$

Let the currents be localized in space and act only for a finite time. Consider a large sphere drawn with the radiating currents near its center, which may also be taken to be the origin of a system of spherical coordinates: Θ, ϕ. An element of surface on the sphere will be $dA = r^2 \sin \Theta \, d\, \Theta \, d\, \phi$. Its normal will be along the radius and outwards: $n = \frac{\hat{r}}{|r|}$. The energy flux at time t through dA per unit time will be

$$\hat{n} \cdot S(rt)dA \tag{4.10}$$

The total energy passing through the fixed element (dA) will be

$$dA\hat{n} \cdot \int_{-\infty}^{\infty} dt\ S(rt) \tag{4.11}$$

Measuring instruments (e.g., the eye) generally respond to the frequency components of the electric and magnetic fields; their contribution to (4.11) may be found by introducing the transforms for E(rt) and B(rt) making use of the Fourier integral theorem:

$$\int_{-\infty}^{\infty} dt\ e^{-\omega t}\ e^{-\omega' t} = 2\pi\ \delta\ (\omega + \omega') \tag{4.12}$$

and noting that for the transform of a real function, f(t), we must have a condition on the complex conjugates:

$$f^*(\omega) = f(-\omega) \tag{4.13}$$

We learn, then, that (4.11) becomes

$$dA\hat{n} \cdot \int_{0}^{\infty} \frac{d\omega}{2\pi}\ \frac{2}{\mu o}\quad \mathrm{Re}\left[E(r\omega)\ x\ B^*\ (r\omega)\right] \tag{4.14}$$

When we realize that $\frac{d\omega}{2\pi} = d\nu$, where $\nu = \omega/2\pi$ is the frequency of the observed light, we can identify

$$dI(\nu) = \frac{2}{\mu o}\ dA\ \mathrm{Re}\ \hat{n}\ \cdot\left[E(r\omega)\ x\ B^*(r\nu)\right] \tag{4.15}$$

with the frequency distribution of the energy flux of the electromagnetic fields through the surface element on the sphere. If dI approaches a finite non-zero limit as the radius of the sphere is increased, the flux may be called radiation, since it is lost energy.

A particle moving uniformly will radiate continuously. To use it in the above scheme requires that we formally treat it as being on for a time T. We take the path as linear, with velocity vector v , passing through the origin at time t = 0:

$$\rho^{ext}(rt) = q\,\delta^{(3)}(r-vt)$$
$$j^{ext}(rt) = qv\,\delta^{(3)}(r-vt) = v\,\rho^{ext}(rt) \tag{4.16}$$

We shall return to (4.6) and (4.7) and rewrite them in the $r\omega$ representation, by means of a convolution in which the singular denominator is isolated.

Using Equation (4.16), we obtain

$$E(r\omega) = i\mu_o\,(\omega v + a^2 i\nabla)\int d^3r'\,G_1(r-r')\rho^{ext}(r'\omega) \tag{4.17}$$

$$B(r\omega) = -\mu_o\,v \times \nabla \int d^3r'\,G_1(r-r')\rho^{ext}(r'\omega) \tag{4.18}$$

where

$$G_1(r-r') = \frac{e^{i\frac{\omega}{a}|r-r'|}}{4\pi|r-r'|} \tag{4.19}$$

As is usual in radiation problems, we take the asymptotic form of G_1. The source point r' is supposed to be near the origin. For large r,

$$|r-r'| \sim |r| - \frac{r}{|r|}\cdot r' = |r| - \hat{n}.r' \tag{4.20}$$

Therefore,

$$G_1(r-r') \sim \frac{e^{i\frac{\omega}{a}|r|}}{4\pi\,|r|}\;e^{-i\frac{\omega}{a}\hat{n}.r'} \tag{4.21}$$

In the evaluation of the fields E and B of Equations (4.17, 18) the gradients are allowed to act only on $e^{i\frac{\omega}{a}r}$, since otherwise the powers of r produced would not be appropriate to a finite contribution to the radiation. Hence, in Equations (4.17, 18) we may replace

$$\nabla \rightarrow i\frac{\omega}{a}\nabla|r| = i\frac{\omega}{a}\hat{n} \tag{4.22}$$

To evaluate the r' integral we write

$$\rho^{ext}(r'\omega) = q\int_{-\infty}^{\infty} dt\, \delta^{(3)}(r' - vt)e^{i\omega t} \tag{4.23}$$

so that

$$\int d^3r'\, e^{-i\frac{\omega}{a}\hat{n},r'}\rho^{ext}(r'\omega) = q\int_{-\infty}^{\infty} dt\, e^{it(\omega - \frac{\omega}{a}\hat{n}.v)}$$

$$= 2\pi q\delta(\omega - \frac{\omega}{a}\hat{n}.v) \tag{4.24}$$

Combining these results leads to the radiation fields

$$E(r\omega) = i\mu_o q\omega a(\hat{n} - \frac{v}{a})\frac{e^{i\frac{\omega}{a}r}}{4\pi r} 2\pi\delta\left[\omega(1 - \hat{n}.\frac{v}{a})\right] \tag{4.25}$$

and

$$B(r\omega) = i\mu_o q\frac{\omega}{a} v \times \hat{n}\frac{e^{i\frac{\omega}{a}r}}{4\pi r} 2\pi\delta\left[\omega(1 - \hat{n}.\frac{v}{a})\right] \tag{4.26}$$

which contribute to the power flux

$$\mathrm{Re}\ \frac{2}{\mu_o}\ E(r\omega)\ \times\ B^*(r\omega)\ =\frac{\mu_o q^2\omega^2}{3\pi|r|^2 a}\ \hat{n}(v^2-\ a^2)\ \delta\ \omega(1-\ \frac{n.v}{a})\ \ 2\pi\delta(0) \tag{4.27}$$

The infinite quantity $2\pi\delta(\omega-\omega) = 2\pi\delta(0)$. arose from the square of the δ- function in E, which in turn arose from the time-integral in Equation (4.23). The use of a source which is on for all times must lead to an infinite amount of power radiated. Only the <u>rate</u> of radiation is physically significant in such a case. Formally, we can resolve the difficulty if we let the source be on only in the time interval -T/2 to T/2, where T is chosen arbitrarily large. Then the integral in Equation (4.24) is still $2\pi\delta(0)$ to sufficient accuracy, but the value of $2\pi\delta(0)$ at its singularity is only the overall time T. With this understanding

$$2\pi\delta(0)\ =\ T \tag{4.28}$$

The <u>rate</u> of <u>radiation</u> becomes

$$dR\ =\ \frac{d\omega}{2\pi}\ \omega\ \frac{\mu_o q^2}{4\pi a}\ (v^2-\ a^2)\delta(1\ -\ \frac{|v|}{a}\ \cos\ \theta)\ \sin\ \theta d\theta d\phi \tag{4.29}$$

in which we have chosen the vector v along the polar axis of spherical coordinates, and have used

$$\delta(ax)\ =\ \frac{1}{|a|}\ \delta(x) \tag{4.30}$$

Using it again,

$$dR\ =\ \frac{\mu_o}{2}\ q^2\ d\nu\ \ \nu\frac{v^2-a^2}{|v|}\ \delta(\cos\ \theta\ -\ \frac{a}{|v|})\ d(\cos\ \theta)\ d\phi \tag{4.31}$$

So, unless v is greater than a (the phase velocity in the medium at frequency $\nu = \frac{\omega}{2\pi}$) there is no radiation at all. The total radiation per unit time is

$$R = \pi\mu_o q^2 \int d\nu \ \nu \frac{v^2 - a^2(\nu)}{|v|} \tag{4.32}$$

the integral being over those frequencies for which $v > a\ (\nu)$. All of the radiation (at frequency ν) lies in the surface of a cone of angle $\Theta = \cos^{-1}(\frac{a}{|v|})$ around the direction of the particle. A transparent substance is capable of supporting visible Cerenkov light by virtue of its being transparent. The efficiency of the process is a consequence of the high values of ν^2 in the optical region. As we shall see, the corresponding process in the plasma behaves differently at low frequencies

CHAPTER V

THE MAGNETO GAS DYNAMIC ANALOGUE OF CERENKOV RADIATION

Let us now proceed to determine the total radiation produced by a single point charge moving with uniform velocity v through a partially ionized, infinite, anistropic plasma, in the direction of an external magnetic field. The plasma under consideration is again the upper region of the earth's ionosphere. The point-charge excitation might be one of the countless hydrogen ions which are emitted near the sun; it might traverse inter-planetary space in corpuscular streams, and arrive at the terrestrial exosphere with speeds of 100 km/sec to 1,000 km/sec. in this region. Therefore, one might expect the uniformly moving charge to radiate in a manner analogous to the Cerenkov effect.

In the normal Cerenkov case, the electric field produced by the particle lies within a cone given by $\cos \Theta = a/v$ around the direction of the particle, and all of the radiation lies in the surface of this cone. As was shown previously in the calculation of the Green's function G_3, the plasma, for a certain mode of oscillation, behaves like a conical wave-guide with propagation allowed within a cone described by

$$\tan \Theta = \frac{1}{\sqrt{\gamma}} = \frac{\omega}{\omega_p} \frac{c}{a} << 1.$$

This cone, which we shall call the Alfven cone, arises because of the basic anisotropy of the medium and is a pure magnetohydrodynamic effect. In general, for $v > a$, the Cerenkov cone lies outside of the Alfven cone when the two vertices coincide. However, when the particle velocity becomes very near the phase velocity, the two cones will coincide and one might expect to find another mode of Cerenkov radiation which propagates in the conical waveguide, above and beyond the analogy to the normal Cerenkov effect.

CALCULATION OF THE ELECTRIC FIELD

Both collisional and thermal damping are negligible for ultra-low frequency magnetohydrodynamic waves propagating along the lines of the earth's magnetic field, for the collision frequencies and temperatures encountered in the exosphere. The latter case will be discussed in detail in the next chapter. Therefore, we can use the collisionless, zero-temperature Green's functions, introduced in Chapter III, to determine the radiation produced by a uniformly moving point charge as described above.

The physically observable quantity, the frequency components of the electric field, can be written as a convolution of the external currents and the various Green's functions

$$E(r\omega) = \lambda\mu_o\omega \int d^3r' \, A(r-r',\omega) \cdot j^{ext}(r',\omega) \tag{5.1}$$

j^{ext} is the current produced by the uniformly moving point charge. Such a point charge, moving in the z-direction with velocity v is described by

$$j^{ext}(rt) = gv\hat{z}\delta^{(3)}(r-vt) = qvz\delta(x)\;\delta(y)\;\delta(z-vt) \tag{5.2}$$

Then $j^{ext}(r\omega)$ is obtained by use of the Fourier integral theorem

$$j(r\omega) = \int e^{i\omega t}\, j(rt)dt = q\hat{z}\;\delta(x)\;\delta(y)\;\exp\left(i\,\frac{\omega}{v}\,z\right) \tag{5.3}$$

Thus from Equation (5.1)

$$E_i(r\omega) = i\mu_o\omega q \int dz' \, A_{iz}(x,y,z-z',\omega)\;\exp\left(i\,\frac{\omega}{v}\,z'\right) \tag{5.4}$$

where

$$A_{xz} = -\frac{c^2}{\omega_p^2}\frac{\partial^2}{\partial x \partial z} G_3 \tag{5.5}$$

$$A_{yz} = -\frac{c^2}{\omega_p^2}\frac{\partial^2}{\partial y \partial z} G_3 \tag{5.6}$$

$$A_{zz} = G_2 - \frac{c^2}{\omega_p^2}\frac{\partial^2}{\partial z^2} G_3 - \left(1 + \frac{1}{\gamma}\right)\frac{\partial^2}{\partial z^2} G_4 \tag{5.7}$$

and

$$\gamma = \frac{\omega_p^2}{\omega^2}\frac{a^2}{c^2} \tag{5.8}$$

G_2, G_3, and G_4 are the zero-temperature collisionless Green's functions discussed at length in Chapter III.

$G_2(r\omega)$ is the exponentially damped Debye-Yukawa potential,

$$\frac{\exp\left(-\frac{\omega_p}{c}|r|\right)}{4\pi |r|}$$

which can be neglected insofar as radiation effects are concerned. Normally, this term would be important for the case of the static charge, but at zero-temperature, the static electric field is also zero.

Making use of the result

$$\begin{aligned}\int dz' \frac{\partial}{\partial z} f(z-z') e^{i\frac{\omega}{v}z'} &= -\int dz' e^{i\frac{\omega}{v}z'} \frac{\partial}{\partial z'} f(z-z') \\ &= i\frac{\omega}{v}\int dz' f(z-z') e^{i\frac{\omega}{v}z'}\end{aligned} \tag{5.9}$$

[with the integrated terms vanishing because we demand that $f(z-z') = 0$ at $z' - \pm\infty$], the electric field can be rewritten

$$E(r,\omega) = i\mu_o\omega q \left(-\frac{c^2}{\omega_p^2}\, i\,\frac{\omega}{v}\right) \nabla \int dz'\, G_3(xyz-z')e^{i\frac{\omega}{v}z'}$$

(5.10)

$$+ \; i\mu_o\omega q \;\frac{\gamma + 1}{\gamma}\,\frac{\omega^2}{v^2}\;\hat{z}\int dz'\, G_4(xyz-z')e^{i\frac{\omega}{v}z'}$$

$$= \mu_o\,\frac{\omega^2}{\omega_p^2}\,\frac{c^2}{v}\,q\nabla\phi + E^{(4)}\hat{z} \tag{5.11}$$

where

$$\phi = \int dz'\, G_3(xyz-z')e^{i\frac{\omega}{v}z'} \tag{5.12}$$

is that part of the field due to G_3, and $E^{(4)}$ is that part of the field due to G_4. Furthermore, since $i\omega B = \nabla x E$ from Maxwell's equations, and the curl of a gradient is zero, B is determined entirely from $E^{(4)}$.

Our first step is therefore to calculate $E^{(4)}$. The Green's function G_4 can be written as a convolution of G_1 and G_3.

$$G_4(xyz-z') = \int dr''\, G_1(r-r'')\, G_3(r''-r')dr'' \tag{5.13}$$

since in wave-number-frequency space, G_4 is just the product of the other two, by definition.

We shall use to best advantage, an integral representation for G_3 that was developed earlier, rather than its explicit form which denotes propagation inside and damping outside of a cone of angle $\theta = \tan^{-1}\frac{1}{\sqrt{\gamma}}$

$$G_3(x''y'',z''-z') = -\frac{i^{-1/2}}{8\pi^{3/2}} \int_0^{\infty} \frac{ds}{s^{3/2}} \exp i \left[\frac{\omega^2}{a^2} s + \frac{(z''-z')^2 - r''^2_T \gamma}{4s} \right] \tag{5.14}$$

then

$$E^{(4)} = -\frac{i^{1/2}\mu_o}{8\pi^{3/2}} \frac{\omega^3}{v^2} \frac{\gamma+1}{\gamma} \hat{z} \int_0^{\infty} \frac{ds}{s^{3/2}} \int dz' \int dr'' \exp i \left[\frac{\omega^2}{a^2} s + \frac{(z''-z')^2 - r''^2_T \gamma}{4s} \right]$$

$$x \quad \exp(i \frac{\omega}{v} z) G_1(r-r'') \tag{5.15}$$

Performing the z' integration first

$$\int_{-\infty}^{\infty} \exp i \frac{\omega}{v} z' \exp\left[\frac{i}{4s}(z''-z')^2\right] = \exp(i \frac{\omega}{v} z'') \exp(- \frac{i\omega^2}{v^2} s)(4\pi is)^{1/2} \tag{5.16}$$

G_1 is the isotropic radiation propagator

$$G_1(r-r'') = \frac{\exp.\ (i \frac{\omega}{a} |r-r''|)}{4\pi|r-r'|} \tag{5.17}$$

where r is the point at which the field is measured and r'' is the source point, relative to a suitable chosen origin.

As we are interested in the fields at large r and in the radiation through a sphere at infinity, let us take the asymptotic form of G_1 for the case $|r| >> |r''|$. Then, Equation (5.17) becomes

$$G_1(r-r')(\text{large } r) = \frac{\exp(i\,\frac{\omega}{a}\,r)\ \exp(-i\,\frac{\omega}{a}\,\hat{n}\,.\,r'')}{4\pi|r|} \tag{5.18}$$

where $\hat{n}$ is a unit vector in the direction of $\vec{r}$. Then

$$E^{(4)} = \frac{\mu_o\, q}{16\pi^2}\ \frac{\gamma+1}{\gamma}\ \frac{\omega^3}{v^2}\ \frac{\exp(i\,\frac{\omega}{a}\,|r|}{r}\ \hat{z}$$

$$x \int_0^\infty \frac{ds}{s} \int dz''dx''dy''\ \exp\ i\left[\left(\frac{\omega^2}{a^2} - \frac{\omega^2}{v^2}\right)s - \frac{r_T''\gamma}{4s}\right] \exp\ i\left[\frac{\omega}{v}\,z'' - \frac{\omega}{a}\,\hat{n}\,.r''\right] \tag{5.19}$$

Next perform the z'' integration which leads to a δ-function

$$\int dz''\ \exp.\ i\left[\frac{\omega}{v}\,z'' - \frac{\omega}{a}\,n_z z''\right] = 2\pi\delta\left(\frac{\omega}{v} - \frac{\omega}{a}\,\hat{n}_z\right) \tag{5.20}$$

where $\hat{n}_z$ is the z-component of the unit vector $\hat{n}$ by z. The x'' and y'' integrals are similar

$$\int dx''\ \exp\left[-i\left(\frac{\gamma}{4s}\,x''^2 - \frac{\omega}{a}\,\hat{n}_x x''\right)\right] = \left(\frac{4\pi s}{i\,\gamma}\right)^{1/2} \exp\left(i\,\frac{\omega^2}{a^2}\,\hat{n}_x{}^2\,\frac{s}{\gamma}\right) \tag{5.21}$$

The y'' integral yields a similar result with $\hat{n}_x{}^2$ replaced by $\hat{n}_y{}^2$. Substituting the results of these three integrations into Equation (5.19), we obtain

$$E^{(4)} = -\frac{i\mu_o\omega^3}{2v^2}\, q\, \frac{\gamma+1}{\gamma}\, \frac{\exp(i\,\frac{\omega}{a}|r|)}{|r|}\, \hat{z}\int_0^\infty ds\, \exp is\left\{ \left(\frac{\omega^2}{a^2} - \frac{\omega^2}{v^2}\right) + \frac{\omega^2}{a^2}\, \frac{(\hat{n}_x{}^2 + \hat{n}_y{}^2)}{\gamma} \right\}$$

$$\delta\left(\frac{\omega}{v} - \frac{\omega}{a}\,\hat{n}_z\right) = \frac{\mu_o q\omega^3}{2v^2}\, \frac{\gamma+1}{\gamma}\, \hat{z}\, \frac{\exp(i\,\frac{\omega}{a}|r|)}{|r|}\, \frac{\delta\left(\frac{\omega}{v} - \frac{\omega}{a}\cos\theta\right)}{\frac{\omega^2}{a^2}\left(1 + \frac{\sin^2\theta}{\gamma}\right) - \frac{\omega^2}{v^2}} \qquad (5.22)$$

where we have replaced $\hat{n}_z$ and $\hat{n}_x{}^2 + \hat{n}_y{}^2$ by $\cos\theta$ and $\sin^2\theta$ respectively. Equation (5.22) gives the electric field due to the term containing G_4. It lies entirely in the z-direction and falls of as $\frac{1}{|r|}$ like a typical radiation field. It should be noted that the only approximation used in obtaining this result was to replace the isotropic propagator G_1 by its asymptotic value for large r since we are concerned with field measurements taken at points a great distance from the origin (or source). Otherwise, the calculation is exact.

Next, we must calculate the scalar potential ϕ which represents that part of the electric field ;due to the Green's function G_3.

$$\phi = \int dz'\, G_3(xyz-z')\exp(i\,\frac{\omega}{v}\,z') \qquad (5.23)$$

Again, using the integral expression for G_3 we can perform the z' integral to give

$$\phi = \frac{(4\pi i)^{1/2}}{8\pi^{3/2}}\, \exp(i\,\frac{\omega}{v}\,z) \int_0^\infty ds\, \frac{\exp\left[i\left(\frac{\omega^2}{a^2} - \frac{\omega^2}{v^2}\right) s\right] \exp\left(-i\,\frac{r_t{}^2\gamma}{4s}\right)}{s} \qquad (5.24)$$

Leaving ϕ in this integral representation, the electric field can now be written as

$$E(r\omega) = \frac{i^{1/2}\mu_o q}{4\pi} \frac{\omega^2}{\omega_p^2} \frac{c^2}{v} \nabla \left\{ \exp(i \frac{\omega}{v} z) \int_o^\infty \frac{ds}{s} \exp\left[i(\frac{\omega^2}{a^2} - \frac{\omega^2}{v^2}) \; s \right] \exp(- \frac{i r_T^2 \gamma}{4s}) \right\}$$

$$+ \frac{\mu_o q \omega^3}{2v^2} \frac{\gamma + 1}{\gamma} \hat{z} \frac{\exp(i \frac{\omega}{a} |r|)}{|r|} \frac{\delta(\frac{\omega}{v} - \frac{\omega}{a} \cos \theta)}{\frac{\omega^2}{a^2}(1 + \frac{\sin^2\theta}{\gamma}) - \frac{\omega^2}{v^2}} \tag{5.25}$$

CALCUALTION OF THE MAGNETIC FIELD

The magnetic field is obtained from the electric field by use of the Maxwell equation

$$B = \frac{\nabla \times E}{i\omega} \tag{5.26}$$

As mentioned earlier, only $E^{(4)}$ contributes to B because the curl of a gradient is zero.

Therefore,

$$i\omega B = \nabla \times \hat{z}\psi \tag{5.27}$$

where ψ is the magnitude of $E^{(4)}$. But using the well known vector identity

$$\nabla \times \hat{z}\psi = - \hat{z} \times \nabla\psi \tag{5.28}$$

and keeping only the leading or $1/|r|$ term after differentiating (which brings down a factor of $\frac{i\omega}{a}$), we obtain

$$B = - \hat{z} \times \frac{\hat{n}\psi}{a} \tag{5.29}$$

where $\hat{n}$ is a unit vector in the r-direction. Explicitly

$$\vec{B} = \frac{\mu_o q\omega^3}{2v^2 a} \frac{\gamma + 1}{\gamma} \frac{\exp(i \frac{\omega}{a}|r|)}{|r|} \frac{\delta(\frac{\omega}{v} - \frac{\omega}{a} \cos \theta)}{\frac{\omega^2}{a^2}(1+ \frac{\sin^2\theta}{\gamma}) - \frac{\omega^2}{v^2}} (+\hat{x}n_y - \hat{y}n_x) \qquad (5.30)$$

where $\hat{x}$ and $\hat{y}$ are unit vectors in the respective directions.

CALCULATION OF THE TOTAL POWER

The Poynting vector $S(r\omega)$ which denotes the total energy radiated per unit area is constructed from the electric and magnetic fields according to the prescription

$$S(r\omega) = E(r\omega) \times B^*(r\omega) \qquad (5.31)$$

The magnetic field is entirely due to the Green's function G_4. The electric field consists of terms involving G_3 and G_4, namely ϕ and ψ respectively. Therefore, the Poynting vector will contain terms involving the products G_3G_4 and G_4G_4. We shall show that the total power due to the terms involving G_3G_4 vanishes over a surface at infinity and that only the $E^{(4)}$ part of the electric field contributes to the radiation. The Green's function G_3 represents propagation inside and damping outside of the Alfven cone of small angle $\theta = \tan^{-1} \frac{\omega}{\omega_p} \frac{c}{a}$. We expect that only the z-component of the part of the Poynting vector involving G_3 will contribute to the total power, since the transverse components lying wholly outside the conical waveguide will be completely attenuated at large enough distances. Therefore, we will construct S_z, the z-component of the Poynting vector, which is independent of terms involving the product G_4G_4, integrate this expression over the infinite plane parallel to the xy plane, at $z=z_o$ and then show that this latter quantity vanishes as $z_o \rightarrow \infty$

$$S_z = E_x B_y^* - E_y B_x^* = \frac{\mu_o^{2}}{\text{[illegible]}} \frac{\gamma + 1}{\gamma^2} \frac{\omega^5}{v^3 a} \frac{c^2 q^2}{\omega_p^{\,2}} \frac{e^{i \frac{\omega}{v} z}}{\frac{\omega^2}{a^2}(1+ \frac{\sin^2\theta}{\gamma}) - \frac{\omega^2}{v^2}}$$

$$\frac{\exp(- i \frac{\omega}{a}|r|)}{|r|^2} (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) \int_0^\infty \frac{ds}{s} \exp\left[i(\frac{\omega^2}{a^2} - \frac{\omega^2}{v^2}) s \right]$$

$$\exp(- \frac{i r_T^{\,2} y}{4s}) \tag{5.32}$$

$dR\omega = \frac{2}{\mu_o} \int n_z .\ S\ dx\ dy$ is the total energy radiated per unit frequency through an infinite plane parallel to the xy plane at $z=z_o$.

$$dR_\omega = \text{const.}\ \exp.(i \frac{\omega}{v} z) \int dx\ dy \int_0^\infty \frac{ds}{s} \frac{\delta(\frac{\omega}{v} - \frac{\omega}{a} \cos\theta)}{\frac{\omega^2}{a^2}(1+ \frac{\sin^2\theta}{\gamma}) - \frac{\omega^2}{v^2}}$$

$$\frac{\exp.(-i \frac{\omega}{a}|r|)}{|r|^2} (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) \exp\left[i(\frac{\omega^2}{a^2} - \frac{\omega^2}{v^2}) s \right]$$

$$\exp.(- \frac{i r_T^{\,2}}{4s}) \tag{5.33}$$

The δ-function imposes a condition on θ. Let us make the double change of variable from $x,y \longrightarrow r,\phi$ and then another change from $r \longrightarrow r_T$ where of course

$$x = r \cos\phi \sin\theta \tag{5.34}$$

$$y = r \sin \phi \sin \theta \tag{5.35}$$

$$r = \sqrt{r_T^{\,2} + z_o^{\,2}} \tag{5.36}$$

and $dxdy \longrightarrow rdrd\phi \longrightarrow r_T dr_T d\phi$ (5.37)

Furthermore

$$x \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \longrightarrow r_T \frac{\partial}{\partial r_T} \tag{5.38}$$

There is no azimuthal angle dependence and the integration of ϕ yields 2π.

Manipulating the δ-function which now involves r_T (since $\cos \theta = \frac{z_o}{\sqrt{r_T^{\,2} + z_o^{\,2}}}$) according to the formula

$$\delta(f(x)) = \frac{\delta(x-x_o)}{\left(\frac{\partial f}{\partial x}\right)_{x_o}} \qquad \text{where } f(x_o) = 0 \tag{5.39}$$

leads to

$$dR_\omega = \text{const. } x \exp.(i \tfrac{\omega}{v} z_o) \int_0^\infty ds \int dr_t \frac{\delta(r_t - r_t^{\,o})}{\frac{\omega^2}{a^2}\left(1 + \frac{r_t^{\,2}}{(r_t^{\,2}+z_o^{\,2})}\right) - \frac{\omega^2}{v^2}}$$

$$\frac{\exp(-i \frac{\omega}{a}\sqrt{z_o^{\,2}+r_T^{\,2}})}{z_o^{\,2} + r_t^{\,2}} \; \frac{(z_o^{\,2}+r_T^{\,2})^{3/2}}{z_o r_T} \; r_T^{\,3} \exp \frac{(i\nu^2 s)}{s^2} \exp.(-i\frac{r_T^{\,2}}{4s})$$

(5.40)

where

$$r_T{}^o = z_o\left(\frac{v^2}{a^2} - 1\right)^{1/2} \tag{5.41}$$

and

$$\nu^2 = \frac{\omega^2}{a^2} - \frac{\omega^2}{v^2} \tag{5.42}$$

Therefore

$$dR_\omega = \frac{\mu_o}{4}\frac{\omega^3}{\omega_p{}^2}\frac{c^2}{\gamma a}\frac{q^2}{v^3}\exp(i\,\frac{\omega}{v}\,z_o)\;z_o{}^2\int_0^\infty \frac{ds}{s^2}\exp(i\nu^2 s)\;\exp(-\frac{i\,z_o{}^2}{4s}(\frac{v^2}{a^2}-1) \tag{5.43}$$

We are primarily interested in the dependence of the total power on z_o, for we wish to show that the above expression will vanish as we remove the plane to infinity.

The s-integral is of the form

$$\int_0^\infty \frac{ds}{s^2}\exp(i\nu^2 s)\;\exp.\;(-\frac{i\delta^2}{s}) = \int dt\;\exp(-i\delta^2)\exp(\frac{i\nu^2}{t}) \tag{5.44}$$

where the change of variable $s = \frac{1}{t}$ has been made and

$$\delta^2 = \frac{z_o{}^2}{4}\left(\frac{v^2}{a^2} - 1\right) \tag{5.45}$$

Now effect another change of variable

$$\delta^2 t = x \tag{5.46}$$

and the integral becomes

$$\frac{1}{\delta^2} \int_0^{\infty} dx \exp(-\, ix) \exp.(\frac{i\nu^2\delta^2}{x}) \tag{5.47}$$

The $\frac{1}{\delta^2}$ in front of the integral contains a $\frac{1}{z_o^2}$ which cancels the z_o^2 in Equation (5.43)

Thus, the entire z_o dependence of the total power (except for the harmonic term) is contained in the integral in (5.47)

If the path of integration is extended into the complex plane to include the infinite quarter-circle in the lower-right hand plane, there is no added contribution to the integral along this path since if x is replaced by x-iy, y positive, exp(-ix) $\longrightarrow$ exp(-ix) exp(-y) which approaches zero as y $\longrightarrow \infty$, and the other term in the integrand is bounded. This argument is analogous to that given for extending the contour when evaluating the integral expression for G_2 in ChapterII. Then, since there are no poles contained within the path of integration, we can deform the contour and integrate along the negative imaginary axis. This is equivalent to replacing x by -iy and integrating from o to ∞. Thus, the integral becomes

$$-i \int_0^{\infty} dy \exp(-\, y) \exp(-\, \frac{\nu^2\delta^2}{y}) \tag{5.48}$$

from which it is readily seen that the expression tends toward zero as $\nu^2\delta^2$ gets very large. For if $\nu^2\delta^2 = 0$, the integral has the value 1 x, and as $\nu^2\delta^2$ starts to grow, it diminishes the y = 0 contribution from exp(-y). But δ^2 is proportional to z_o^2, and thus as $z_o \longrightarrow \infty$, both the integral in Equation (5.48) and the expression for dR_ω the remainder of which is independent of z_o, tend to zero.

Therefore, that part of the Poynting vector which contains the product G_3G_4, when integrated over an infinite plane, parallel to the x-y plane, at z_o, gives no contribution to the total power when the plane is taken at very large distances from the source. Thus, the only part of the Poynting vector contributing to the total radiated power is that which contains the product $G_4 G_4$.

The remaining portion of the Poynting vector, which we shall denote as $S^{(4)}$, results in a uniform flux through a sphere even when the radius of the sphere is infinite. This flux may be considered radiation, since it is energy lost to the system. As we shall see, the total power radiated is a rather small quantity, but it seems to be the dominant process for removing energy from the incoming particle, since the power lost by synchrocyclotron radiation at nonrelativistic velocities is negligible, and the Bremsstrahlung effect is absent.

$S^{(4)}$, which depends on $E^{(4)}$ and B, contains a double δ-function. Then, as was discussed in Chapter V, the integral over the double δ-function yields a time T, which is the value of $2\pi\delta(0)$, accounting for the fact that the current source is not on for all time but over an arbitrarily large time interval T.

$$S^{(4)} = E^{(4)} \times B^* = - \frac{\mu_o^2}{4} \frac{(\gamma+1)^2}{\gamma^4} \frac{\omega^6}{av^2} \frac{\delta(\omega-\omega\frac{v}{a}\cos\theta)\ \delta(\omega-\omega\frac{v}{a}\cos\theta)}{\left[\frac{\omega^2}{a^2}(1+\frac{\sin^2\theta}{\gamma}) - \frac{\omega^2}{v^2}\right]^2}$$

$$\frac{q^2}{r^2}\hat{z} \times (\hat{z} \times \hat{n}) \tag{5.49}$$

and integrating over a sphere of radius r gives the total energy radiated

$$dI_\omega = \frac{2}{\mu_o}\int \hat{n} \cdot S^{(4)} dA$$

$$= \frac{\mu_o}{2} \frac{\omega^6}{av^2} q^2 \frac{(\gamma+1)^2}{\gamma^4} \frac{T}{2\pi} \int \frac{\delta(\omega-\omega\frac{v}{a}\cos\theta)\ \sin^2\theta\ d(\cos\theta)d\phi}{\left[\frac{\omega^2}{a^2}(1+\frac{\sin^2\theta}{\gamma} - \frac{\omega^2}{v^2}\right]^2}$$

$$(5.50)$$

There is no azimuthal dependence and the ϕ-integration yields 2π.

T is the actual value of $\delta(0)$ and represents the time interval over which the current acts.

Since $S^{(4)}$ is a function of r and ω, dI_ω represents the total energy radiated rather than the total power. To obtain the latter, we must divide the above expression by the time interval T. Thus,

$$dR_\omega = \frac{dI_\omega}{T} = \frac{\mu_o}{2} \frac{\omega a^4}{v^3 \gamma^2} \frac{1}{2} \frac{\delta(\cos \Theta - \frac{a}{v})}{1 - \frac{a^2}{v^2}} d(\cos \Theta \tag{5.51}$$

and

$$dR = \frac{\mu_o}{2} \frac{c^4}{v^3} \frac{q^2}{\omega_p^4} \omega^5 d\omega \frac{\delta(\cos \Theta - \frac{a}{v})}{1 - \frac{a^2}{v^2}} d(\cos \Theta) \tag{5.52}$$

Again, as in the normal Cerenkov effect, there is no radiation unless v is greater than $a(\omega)$. Then, all of the radiation lies in the surface of a cone of angle $\Theta = \cos^{-1} \frac{a}{v}$ around the direction of the particle.

The total radiation per unit time is

$$R = \frac{\mu_o}{2} \frac{c^4}{v^3} \frac{q^2}{\omega_p^4} \int_o^\omega \frac{\omega^5 d\omega}{1 - \frac{a^2(\omega)}{v^2}} \tag{5.53}$$

Here, the total radiated power varies as the sixth power of the frequency as opposed to the normal Cerenkov case when the frequency dependence went as ω^2. Furthermore, the factor of $v^2 - a^2$, which appeared in the numerator in the normal Cerenkov case now is found in the denominator. This suggests that the radiation becomes arbitrarily large when $v \rightarrow a$. But, when $v=a$, $\cos \Theta = 1$ and the radiation would be directed along the magnetic field lines. From the symmetry of the problem, one expects the radiation to fall to zero along the direction of the incoming particle. We must look more closely at the fields to determine the correct expression for small angles, Θ, in the limit of $v \longrightarrow a$.

In the calculation of $E^{(4)}$, the asymptotic, large $|r|$, far-field expression for G_1 was used. This asymptotic form is valid only if the source term mutiplying G_1 in the integrand is localized. In the normal Cerenkov effect, the source term is merely the current, $j^{ext} = qv\delta(z-vt)$ which is indeed localized. However, in the present case, the source term is the product $G_3 j$ which is not localized since G_3 is singular on the surface of the Alfven cone. Thus, the asymptotic approximation for G_1 should be valid at large distances provided that we remain also outside of the cone. These expressions are incorrect for very small angles lying outside the Alfven cone and an alternative method must be used.

Returning again to the expression for $E^{(4)}$, Equation (5.10), let use for G_4 the integral expression developed in Chapter III.

$$G_4(xyz) = \frac{-i^{-3/2}}{8n^{3/2}} \int_0^\infty \frac{du}{u^{1/2}} \exp(i \frac{\omega^2}{a^2} u + i \frac{z^2}{4u})$$

$$\int_{-u/\gamma}^{u} \frac{d\zeta}{\zeta} \exp(i \frac{r_T^{\ 2}}{4\zeta}) \frac{\gamma}{1+\gamma}$$

$$E^{(4)} = -iu_o \omega z \left(\frac{\gamma+1}{\gamma}\right) \frac{\partial^2}{\partial z^2} \int G_4(z-z') \,.\, j(z')dz$$

$$= \frac{i^{-1/2}\mu_o}{8\pi^{3/2}} q\omega\hat{z} \int_0^\infty \frac{du}{u^{1/2}} \exp(i \frac{\omega^2}{a^2} u + i \frac{(z-z')^2}{4u}) \int_{-u/\gamma}^{u} \frac{d\zeta}{\zeta} \quad x$$

$$\exp(i \frac{r_T^{\ 2}}{4\zeta}) \int_{-\infty}^{\infty} e^{i\omega t} \delta(z'-vt) \, dz'dt \tag{5.54}$$

and

$$B = \frac{\nabla x E^{(4)}}{i\omega} = -\frac{z x \nabla E^{(4)}}{i\omega} \tag{5.56}$$

It will be more convenient here to work with B rather than E. B_x and B_y will involve $\frac{\partial E^{(4)}}{\partial y}$ and $\frac{\partial E^{(4)}}{\partial x}$ and the taking of these derivatives will enable us to do the ζ-integral and also the u-integral.

$$B_x = -\frac{i}{\omega}\frac{\partial}{\partial y} E^{(4)} = -\frac{2iy}{\omega}\frac{\partial}{\partial {r_T}^2} E^{(4)} \tag{5.57}$$

and a similar expression for B_y with y replaced by -x. The only r_T dependence in $E^{(4)}$ is contained in the ζ-integral.

$$\frac{\partial^2}{\partial {r_T}^2}\int_{u/\gamma}^{u}\frac{d\zeta}{\zeta}\exp(i\frac{{r_T}^2}{4\zeta}) = -\frac{1}{{r_T}^2}\left[\exp(i\frac{{r_T}^2}{4u}) - \exp(-i\frac{{r_T}^2\gamma}{4u})\right] \tag{5.58}$$

Denoting the u-integral by I(u), it becomes

$$I(u) = \int_0^\infty \frac{du}{u^{1/2}}\exp(i\frac{\omega^2}{a^2}\omega)\left[(\exp\left[i\frac{(z^2+{r_T}^2)}{4u}\right] - \exp\left[i\frac{(z^2-{r_T}^2\gamma)}{4u}\right]\right] \tag{5.59}$$

By making the substitution $u = 1/x$, each of these integrals can be put in the form

$$\int_0^\infty \frac{dx}{x^{3/2}}\exp(i\frac{A^2}{4}x)\exp(i\frac{B^2}{x}) \text{ whose solution is}$$

$$\sqrt{\pi i}\ \frac{\exp.(iAB)}{B}.$$

Therefore

$$I(u) = \frac{\sqrt{\pi i}}{\frac{\omega}{a}} \quad \exp(i \frac{\omega}{a} \sqrt{(z-z')^2+r_T^2}) - \exp(i \frac{\omega}{a} \sqrt{(z-z')^2-r_T^2\gamma}) \tag{5.60}$$

Then, from Equations (5.60),(5.57) and (5.55)

$$B_x = -\frac{\mu_o}{4\pi} q \frac{y}{r_T^2} \frac{a}{\omega} \frac{\partial^2}{\partial z^2} \int_{-\infty}^{\infty} e^{i\omega t} dt \exp(i \frac{\omega}{a}\sqrt{(z-vt)^2+r_T^2})$$

$$- \exp(i \frac{\omega}{a} \sqrt{(z-vt)^2 - r_T^2 \gamma}) \tag{5.61}$$

We now assume that the external current is localized within the Alfven cone, i.e., $v^2t^2 < z^2 - r_T^2$. Then

$$\exp(i \frac{\omega}{a}\sqrt{(z-vt)^2+r_T^2}) \sim \exp(i \frac{\omega}{a}\sqrt{z^2+r_T^2}) \exp(-i \frac{\omega}{a} \frac{vtz}{\sqrt{z^2+r_T^2}}) \tag{5.62}$$

and

$$\exp(i \frac{\omega}{a} \sqrt{(z-vt)^2-r_T^2\gamma}) \sim \exp(i \frac{\omega}{a} \sqrt{z^2-r_T^2\gamma} \exp(-i \frac{\omega}{a} \frac{vtz}{\sqrt{z^2-r_T^2\gamma}}) \tag{5.63}$$

Performing the t-integral, we obtain

$$B_x = -\frac{\mu_o q}{2} \frac{y}{r_T^2} \frac{\omega}{a} \frac{\partial^2}{\partial z^2} \exp(i \frac{\omega}{a}|r|) \; \delta(\omega-\omega \frac{v}{a} \frac{z}{|r|}) - \exp(i \frac{\omega}{a} |\bar{r}|)$$

$$\delta(\omega-\omega \frac{v}{a} \frac{z}{|\bar{r}|} \tag{5.64}$$

where again $|r| = (z^2+r_T^2)^{1/2}$

and $|\bar{r}| = (z^2-r_T^2\gamma)^{1/2}$

The leading term remaining in each expression after taking two derivatives is just the original expression multiplied by the factor $-\frac{\omega^2}{a^2}\,\frac{z^2}{r^2}$ for the term involving r and $-\frac{\omega^2}{a^2}\,\frac{z^2}{\bar{r}^2}$ for the term involving $\bar{r}$. We are interested in the radiation part of the field and hence keep only those terms that exhibit a 1/r dependence.

Therefore

$$B_x = \frac{u_o q}{2}\,\frac{\omega}{a}\,\frac{y}{r_T^2}\,\frac{z^2}{r^2}\left[\exp(i\,\frac{\omega}{a}\,|r|)\;\delta(\omega-\omega\,\frac{v}{a}\,\frac{z}{|r|}) - \frac{z^2}{\bar{r}^2}\exp(i\,\frac{\omega}{a}|\bar{r}|)\right.$$

$$\left.\delta(\omega-\omega\,\frac{v}{a}\,\frac{z}{|\bar{r}|})\right] \tag{5.65}$$

In order to see the angular dependence of the field, let us convert to spherical coordinates, r, θ, ϕ.

The δ-function appearing in the first term can be written

$$\delta(\omega-\omega\,\frac{v}{a}\,\frac{z}{|r|}) = \delta(\cos\theta - \frac{a}{v})\frac{a}{v\omega} \tag{5.66}$$

and the other δ-function can be cast into the form

$$\delta(\omega-\omega\,\frac{v}{a}\,\frac{z}{|\bar{r}|}) = \frac{1}{\omega}\,\delta\left(\cos\theta - \sqrt{\frac{\gamma}{1+\gamma-\frac{v^2}{a^2}}}\right)\frac{v^2/a^2}{(1+\gamma-v^2/a^2)^{3/2}} \tag{5.67}$$

making use of the relation

$$\delta(f(x)) = \frac{\delta(x-x_o)}{\left(\frac{\partial f}{\partial x}\right)_{x_o}}$$

where $f(x_o) = 0$.

Then, making use of Equations (5.66) and (5.67) in (5.65)

$$B_x = \frac{u_o q}{2} \quad \frac{\sin \phi}{\sqrt{v^2 - a^2}} \quad \frac{1}{r} \left[\frac{\exp(i\,\frac{\omega}{a}|r|)\,\delta(\cos\Theta - \frac{a}{v})}{v^2/a^2} \right.$$

$$\left. -\sqrt{\gamma} \quad \frac{\exp(i\,\frac{\omega}{a}\,|\bar{r}|)\ \delta\left(\cos\Theta - \sqrt{\frac{\gamma}{1+\gamma-v^2/a^2}}\right)}{a/v\ \ (1+\gamma-v^2/a^2)} \right] \tag{5.68}$$

The two terms in this expression are mutually exclusive. If one contributes to the magnetic field, the other does not. The first term contributes only for $v/a > 1$, since the argument of the δ-function vanishes when $\cos\Theta = a/v$. However, the second term contributes when $v/a < 1$ as is easily seen from examining its δ-function. Thus, the first expression, resulting from the $\exp(i\ \omega/a|r|)$ term in G_4, is analogous to the normal Cerenkov effect. The magnetic field is contained in a cone of angle $\Theta_c = \tan^{-1}\frac{\sqrt{v^2 - a^2}}{a}$, and occurs only when the particle velocity is greater than the phase velocity. The second term leads to a strikingly new effect which we shall henceforth call the anomalous Cerenkov Effect. Here the radiation field is contained within a cone of angle

$$\underline{\Theta}' = \tan^{-1}\left(\frac{1-v^2/a^2}{\gamma}\right)^{1/2}$$

and occurs only if the particle velocity is <u>less than</u> the phase velocity. The highly anisotropic plasma is capable of sustaining a radiation field regardless of the velocity of the moving charged particle, whereas normal Cerenkov radiation is possible in an isotropic medium only when the moving charge traverses the medium at speeds greater than the phase velocity of light in the medium. Moreover, note that the cone accompanying the anomalous Cerenkov radiation always lies within the Alfven cone and coincides with it in the limit of $v/a \to 0$. Mathematically, this is a consequence of the fact that this new effect arose from the conical waveguide term $\exp(i \frac{\omega}{a} |\bar{r}|)$ appearing in G_4. Previously, this term did not appear because of our inability to treat the Alfven cone properly. Its physical manifestation is of unknown origin. The overall factor of $\gamma^{-1/2}$ multiplying this term partially explains the fact that such radiation has never been detected. For $\omega = 10\sec^{-1}$, the amplitude of the magnetic field associated with the anomalous effect is 10,000 times smaller than the field giving rise to the normal Cerenkov radiation

However, it is evident that our treatment of the problem is not adequate for very small angles. Note that as $\Theta \to 0$, each term becomes arbitrarily large as v approaches a form above and below, respectively. This results because each term is proportional to $\csc \Theta$ which in each case is dependent on the quantity $\frac{1}{\sqrt{v^2-a^2}}$. As will be shown shortly, the expression for the electric field $E^{(4)}$ will also have two terms which are both independent of $\sin \Theta$ and therefore finite as $\Theta \to 0$. However, the product of E and B, in the Poynting vector, will result in a finite, nonzero total power for $\Theta = 0$. But, we expect zero power as we look along the path of the moving charge because there is no transverse direction. Moreover, the Green's Function G_4 is known exactly for the case of $\Theta = 0$ ($r_T = 0$). Thus, we can calculate the exact expressions for the electric and magnetic fields, and check that the power really is zero.

$$G_4(r_T = 0) = \frac{\exp(i \frac{\omega}{a} (z-z'))}{8\pi \frac{\omega}{a}} \frac{\gamma}{1+\gamma} \ln \gamma \tag{5.69}$$

$$E^{(4)}(r_T = 0) = \mu_o \frac{\omega^3}{v^2} q \int G_4(z-z') \exp(i \frac{\omega}{v} z') dz'$$

$$= \frac{\mu_o}{4} \frac{\omega^2}{v^2} \delta(\frac{\omega}{v} - \frac{\omega}{a} \exp(i \frac{\omega}{a} z) \hat{z}. \qquad (5.70)$$

which is not a radiation field since it does not exhibit the classical 1/r behavior, but rather more like a plane wave solution. The magnetic field for small angles can be obtained by returning to Equation (5.61) and expanding $\exp(i \frac{\omega}{a}|r|) - \exp(i \frac{\omega}{a}|\bar{r}|)$ in a Taylor series about small r_T. Then

$$B_x = -\frac{\mu_o q}{4\pi} \frac{a}{\omega} y \frac{\partial^2}{\partial z^2} \int_{-\infty}^{\infty} e^{i\omega t} dt \frac{\exp(i \frac{\omega}{a}|z-vt|)}{|z-vt|} (\gamma+1) \qquad (5.71)$$

which goes to zero as $y \longrightarrow 0$. Likewise, B_y which is proportional to x also vanishes in this limit. The magnetic field thus vanishes for $\theta = 0$ because when treated correctly, the two terms contribute together and oppose each other. In Equation (5.65) the two terms cannot cancel each other, even in the limit of $\theta \longrightarrow 0$, because they do not exist simultaneously, i.e., the existence of one precludes the existence of the other. Yet, when the problem is treated properly for small r_T, we see that the two terms contribute together and the net result is zero. This is to be expected since when $r_T = 0$, $r = \bar{r} = z$ and the two terms are identical. The inconsistency present here is a consequence of our inability to treat the isotropic propagator $\exp(i \frac{\omega}{a}|r|)$ when $|r|$ is inside the Alfven cone and θ is near zero. The mechanism that prevents the field calculated in Equation (5.67) from growing arbitrarily large comes into play when the θ given by $\theta_c = \tan^{-1} \frac{\sqrt{a^2-v^2}}{a}$ lies within the Alfven cone. For it must be noted that even though the medium behaves as a conical waveguide, there is nothing in the normal Cerenkov term to denote our passing from the damping region outside of this cone to the region of propagation inside. We would expect, if the calculation were performed correctly, that the normal Cerenkov term would exhibit a change in character, as Θ is decreased to lie within Θ_a. Bearing in mind the fact that the magnetic field vanishes for $\Theta = 0$ in view of the exact calculation and the heuristic arguments given above, we can use Equation (5.67) to describe the field at all points off the magnetic field axis. Since for ELF waves Θ_a is of the order of 10^{-4} radians, the normal Cerenkov term is correct down to the smallest angles. In addition, since the radiation accompanying the anomalous effect is contained within the Alfven cone, this radiation would appear to be concentrated in infinitesimally thin tubes centered about the magnetic field lines.

In order to determine the electric field $E^{(4)}$, we use still another integral representation of G_4. In Chap. III, we developed an expression for G_4 involving an integral over ${r_T}^2$

$$G_4(z-z') = \left(\frac{-i\gamma}{1+\gamma}\right) \frac{1}{8\pi\frac{\omega}{a}} \int_0^{{r_T}^2} \frac{d(u^2)}{u^2} \left[\exp\left(i \frac{\omega}{a} \sqrt{(z-z')^2+u^2}\right) - \exp\left(i \frac{\omega}{a} \sqrt{(z-z.)^2-\gamma u^2}\right)\right] + i \frac{\exp(i \frac{\omega}{a}|z-z'|)}{8\pi \frac{\omega}{a}} \frac{\gamma}{1+\gamma} \ln\gamma \tag{5.72}$$

$$\begin{aligned} E^{(4)} &= -i\mu_o\omega \frac{\gamma+1}{\gamma} vq\hat{z} \frac{\partial^2}{\partial z^2} \int G_4(z-z')e^{i\omega t}\delta(z'-vt)dzdt \\ &= -\frac{\mu_o va}{8\pi} q\hat{z} \int_0^{{r_T}^2} \frac{d(u^2)}{u^2} \int_{-\infty}^{\infty} e^{i\omega t}dt \left[\exp\left(\frac{i\omega}{a}\sqrt{(z-vt)^2+u^2}\right) - \exp.\left(i \frac{\omega}{a} \sqrt{(z-vt)^2-\gamma u^2}\right)\right] + {E_o}^{(4)}\hat{z} \end{aligned} \tag{5.73}$$

where $E_o^{(4)}$ is the value of the electric field for Θ or $r_T = 0$

$$E_o^{(4)} = \frac{\mu_o}{4} \frac{\omega a^2}{v^2} \delta(1 - \frac{a}{v}) \exp(i \frac{\omega}{a}) \tag{5.74}$$

Again we assume that the source is localized within the Alfven cone, i.e.,

$$v^2t^2 << z^2 - u^2$$

and expand the exponentials as before. This leads to

$$E^{(4)} = \frac{\mu_o v}{4} \; aq\hat{z} \frac{\partial^2}{\partial z^2} \int_o^{r_T^2} d \frac{(u^2)}{u^2} \left[\exp(i \frac{\omega}{a}|r|)\delta(\omega-\omega \frac{v}{a} \frac{z}{|r|}) \right.$$

$$\left. -\exp(i \frac{\omega}{a}|\bar{r}|)\delta(\omega-\omega \frac{v}{a}|\bar{r}|)\right] + E_o^{(4)}\hat{z} \tag{5.75}$$

In order to perform the remaining integrals, we write the δ-functions in the form $\delta(u^2-u_o^2)$

$$\delta(\omega-\omega \frac{v}{a} \frac{z}{\sqrt{z^2+u^2}}) = \delta\left[u^2 - z^2(\frac{v^2}{a^2} -1)\right] 2 \frac{v^2z^2}{a^2\omega} \tag{5.76}$$

$$\delta(\omega-\omega \frac{v}{a} \frac{z}{\sqrt{z^2- u^2\gamma}}) = \delta\left[u^2 - \frac{z^2}{\gamma}(1- \frac{v^2}{a^2})\right] \frac{2v^2z^2}{a^2\omega\gamma} \tag{5.77}$$

From the previous calculation of the magnetic field, we know that the first δ-function contributes only when v is greater than a, and the latter only when v < a. Furthermore, the value of r_T^2 in the limits of the integration must be large enough in each case so that the point where the δ-function is nonzero lies within the limits of the integral. Then

$$E^{(4)} = \frac{-\mu_o}{4} \frac{v^3}{a\omega} q\hat{z} \frac{\partial^2}{\partial_z^2} \exp(i \frac{\omega}{a} \frac{v}{a}|z|) \left[\frac{H\left(r_T^{\,2} - z^2(\frac{v^2}{a} - 1)\right)}{\frac{v^2}{a^2} - 1} - \frac{H\left(r_T^{\,2} - \frac{z^2}{\gamma}(1 - \frac{v^2}{a^2})\right)}{1 - \frac{v^2}{a^2}} \right] E_o^{(4)} \hat{z} \tag{5.78}$$

where

$$H(r_T^{\,2} - z^2(\frac{v^2}{a^2} - 1)) = 1 \tag{5.79}$$

for

$$r_T^{\,2} > z^2(\frac{v^2}{a^2} - 1) \tag{5.80}$$

and zero otherwise, etc. In taking two z-derivatives of the quantity in the brackets, we keep only those terms that have the 1/r dependence required for a radiation field. The terms of interest are the cross terms obtained by differentiating once both the exponentials and the H-functions. The derivative of an H function gives a δ-function of similar argument, i.e.,

$$\frac{\partial}{\partial z} H\left(r_T^{\,2} - z^2(\frac{v^2}{a^2} - 1)\right) = -2z\, \delta\left[r_T^{\,2} - z^2(\frac{v^2}{a^2} - 1)\right] (\frac{v^2}{a^2} - 1) \tag{5.81}$$

$$\frac{\partial}{\partial z} \exp(i \frac{\omega}{a} \frac{v}{a}|z|) = i \frac{\omega}{a} \frac{v}{a} \operatorname{sgn} z \tag{5.82}$$

where sgn z denotes the sign of z.

Furthermore,

$$\delta\left[r_T^{\,2}-z^2\left(\frac{v^2}{a^2}-1\right)\right]=\frac{1}{r^2}\,\delta\left(1-\frac{v^2}{a^2}\frac{z^2}{r^2}\right)=\frac{1}{2r^2}\,\delta\left(1-\frac{v}{a}\frac{z}{r}\right)=\frac{a}{2vr^2}\delta\left(\cos\Theta-\frac{a}{v}\right) \tag{5.83}$$

$$\delta\left[r_T^{\,2}-\frac{z^2}{\gamma}\left(\frac{v^2}{a^2}-1\right)\right]=\frac{\gamma}{r^2}\,\delta\left(1-\frac{v^2}{a^2}\frac{z^2}{r^2}\right)$$

$$=\frac{\gamma}{2r^2}\,\delta\left(\cos\Theta-\sqrt{\frac{\gamma}{1+\gamma-\frac{v^2}{a^2}}}\right)\frac{\sqrt{\gamma}\;v^2/a^2}{(1+\gamma-v^2/a^2)^{3/2}} \tag{5.84}$$

Note that the δ-functions appearing here, when written in spherical coordinates are the same δ-functions contained in the expression for the magnetic field.

Thus, taking the appropriate derivatives and rearranging the resultant δ-functions and converting entirely to spherical coordinates leads to

$$E^{(4)}=\frac{\mu_o}{4}\frac{v^4}{a^3}\,q\hat{z}\,\frac{\exp(i\,\frac{\omega}{a}\,r\,)}{|r|}\left[\frac{\delta(\cos\Theta-\frac{a}{v})}{\frac{a}{v}}-\frac{\delta\left(\cos\Theta-\sqrt{\frac{\gamma}{1+\gamma-\frac{v^2}{a^2}}}\right)e^{i\sigma}}{1+\gamma-\frac{v^2}{a^2}}\right.$$

$$+E^{(4)}\hat{z} \tag{5.85}$$

Again both the analogue to the normal Cerenkov effect and the anomalous Cerenkov term are present in this expression. However, unlike the result derived earlier for the magnetic field, the electric field does not become arbitrarily large as $v \rightarrow a$ from above and/or below but merely approaches a finite value. If Equation (5.85) were entirely correct, the field would approach the value $E^{(4)}(0)$, but, as can be easily seen, the terms involving the δ-functions do not vanish as $v \to a$. As a consequence to the fact that $E^{(4)}$ does not reduce to its true value as $r_T = 0$, the total power will not vanish for this same case, but instead approaches some constant value. Actually, the electric field at $r_T = 0$ is not a radiation field but a plane wave solution of the form $\exp(i \frac{\omega}{a}|z|)$ and hence should not contribute to the total power. The δ-functions terms in Equation (5.85) keep their $1/r$ dependence even in the limit of $r_T \to 0$. The reason for this discrepancy, as stated above, is not known, and is contained in the mathematics. If the electric field depended on $\sin\Theta$, the term $\sqrt{1 - \frac{a^2}{v^2}}$ would appear in the numerator, and the $r_T = 0$ value $(v = a)$ of the field would then reduce to its true plane wave value.

In computing the total power, we shall treat the two effects separately.

1. ANALOGUE TO NORMAL CERENKOV EFFECT - $v \geq a$.

The fields contributing to this effect are obtained from equations (5.85) and (5.68)

$$E^{(4)} = \frac{\mu_o}{4} \frac{v^3}{a^2} q\hat{z} \; \frac{\exp(i \frac{\omega}{a}|r|)}{|r|} \delta(\cos\Theta - \frac{a}{v}) \tag{5.86}$$

$$B = \frac{\mu_o q}{2} \frac{a^2}{v} \exp \frac{(i \frac{\omega}{a}|r|)}{|r|} \; \frac{\delta(\cos\Theta - \frac{a}{v})}{\sqrt{a^2-v^2}} (\sin\phi\,\hat{X} - \cos\phi\,\hat{y}) \tag{5.87}$$

Then $S = E \times B^*$

$$= \frac{\mu_o^2 q^2}{8} \frac{v^2}{|r|^2} \; \frac{\delta(\cos\Theta - \frac{a}{v})}{\sqrt{v^2-a^2}} \delta(\cos\Theta - \frac{a}{v})(\cos\phi\,\hat{X} + \sin\phi\,\hat{Y}) \tag{5.88}$$

Again, one of the δ-functions is identified with T, the overall time that the source is on.

$$2\pi\ \delta(0) = \omega T \tag{5.89}$$

The total energy radiated per unit frequency is given by

$$dI_\omega = \frac{2r^2}{\mu_o}\ \hat{n}.S\ d(\cos\theta)d\phi \tag{5.90}$$

and the rate of radiation is just the above expression divided by T.

The total power radiated by the moving point charge is then obtained by integrating over all solid angles and over the frequency

$$R = \int d(\cos\theta) \int d\phi \int \frac{d\omega}{2\pi}\ \frac{\mu_o q^2}{8}\ \frac{v^2\omega}{\sqrt{v^2-a^2}}\ \delta(\cos\theta - \frac{a}{v})\ \sin\theta$$

$$= \frac{\mu_o \pi v q^2}{4} \int \nu d\nu \tag{5.91}$$

The integral being over those frequencies in the ELF range for which $\nu > a(\nu)$.

Both the total power and the power per unit solid angle are independent of the factor v^2-a^2 and furthermore are constant for a given v.

The normal Cerenkov radiation, as given by Equation (4-32) is

$$R_c = \pi\mu_o q^2 \int d\nu\nu\ \frac{v^2-a^2(\nu)}{|v|} \tag{5.92}$$

which for the case of $v \gg a$, reduces to the expression obtained for the present case (except for a factor of four.)

This result is not too surprising. Our expressions for the fields are most correct in the region far outside of the Alfven cone, where the particle velocity is much greater than the phase velocity. Obviously, the above expression for the total power, Equation (5.91), does not go to zero as $v \rightarrow a$, but approaches a finite constant value. One can conjecture that a factor of

v^2-a^2 in the numerator would suffice to yield an expression compatible with the $\Theta = 0$, $v \longrightarrow a$, case, but there would still be no reference made to the existence of the Alfven cone. In fact, because of the nature of the Alfven cone, one should expect the radiation to be enhanced as the Cerenkov cone first coincides with the former. The radiation should then reach a peak for some value of $\Theta_c < \Theta_a$, and then decrease to zero at $\Theta = 0$. The mathematical mechanism that does this is absent from the expression for the magnetic field. When the ratio of a/v is such that the two cones coincide, the magnetic field, as given by Equation (5.65) is $\sqrt{\gamma}$ times as great as its value for large v ($\frac{v}{a} >> 1$). This represents a large buildup of the field strength. However, as we approach $\Theta_c = 0$, this expression becomes infinite instead of cutting off at some value of Θ_c at inside the Alfven cone. A summary of these results will be presented at the conclusion of this chapter.

2. THE ANOMALOUS CERENKOV EFFECT

Looking now at the fields produced by the anomalous effect, the terms of interest are

$$B = -\frac{\mu_o q}{2}\,\frac{v}{a|r|}\,\frac{\exp(i\,\frac{\omega}{a}|\bar{r}|)}{(1+\gamma-v^2/a^2)}\,\delta\left(\cos\Theta - \sqrt{\frac{\gamma}{1+\gamma-v^2/a^2}}\right)(\sin\phi\,\hat{x}-\cos$$

and

$$E^{(4)} = -\frac{\mu_o}{4}\,\frac{v^4}{a^3}\,q\hat{z}\,\frac{\exp(i\,\frac{\omega}{a}|\bar{r}|)}{|\bar{r}|}\,\frac{\delta\left(\cos\Theta - \sqrt{\frac{\gamma}{1+\gamma-\frac{v^2}{a^2}}}\right)}{1+\gamma-\frac{v^2}{a^2}} \qquad (5.94)$$

where now $v \leq a$. This effect does not occur when a uniformly moving charged particle traverses an <u>isotropic</u> medium. The accompanying Poynting vector is

$$S = \frac{\mu_o q^2}{8}\,\frac{v^5}{a^4}\sqrt{\frac{\gamma}{v^2-a^2}}\,\frac{1}{r^2}\,\frac{\delta(\cos\Theta - b)}{(1+\gamma-v^2/a^2)^2}\,\frac{\omega T}{2\pi}\,(\cos\phi\,\hat{x} + \sin\phi\,\hat{y})$$

5.95)

where again one of the δ-functions has been associated with the time T and

$$b = \sqrt{\frac{\gamma}{1 + \gamma - v^2/a^2}}$$

Then

$$R = \frac{2}{\mu_o} \frac{r^2}{T} \int n \cdot S \; d(\cos \theta) \; d\phi \; \frac{d\omega}{2\pi}$$

$$= \frac{u_o q^2}{4} \pi v^4 \sqrt{\gamma} \int \frac{\nu d\nu}{a^4(\nu)(1 + \gamma - v^2/a^2)^{5/2}} \qquad (5.96)$$

gives the radiation over all frequencies in the ELF range for which $v < a(\nu)$. Note that for all extensive purposes, this expression is proportional to $1/\gamma^2$ and thus is a very small quantity compared to the ordinary Cerenkov result. ($\gamma = 10^8$ for $\omega = 10\ \mathrm{sec}^{-1}$) Thus the radiation for all values of $v < a$ is contained in the surface of a cone of angle $\theta'_c \approx 10^{-4}$ radians and is of very low intensity. For $v/a \approx 1$, the radiation from a single charged particle for all ELF frequencies up to 100 cps is of the order of 10^{-56} watts. Even a beam of particles, each contributing collectively, would not raise this contribution to a value that could be detected above the background noise. In fact, it is highly doubtful whether either the radiation or the associated magnetic field could be detected by any conventional means, assuming such an attempt was made.

SUMMARY OF RESULTS

A uniformly moving charged particle, traversing the ionosphere in the direction of the lines of force of the earth's geomagnetic field, produces low intensity radiation at ELF frequencies. If the particle velocity is greater than the phase velocity, the radiation is contained in the surface of a cone of angle $\theta_c = \cos^{-1}(a/v)$.

For large values of v/a, the radiation and the associated fields are nearly identical with those produced by the normal Cerenkov effect which is observed in isotropic media. As v approaches a from above, the magnetic field builds up to very large values and reaches a maximum for some value of $\frac{a}{v} < \sqrt{\frac{\gamma}{\gamma+1}}$. Because of improper treatment of the problem, the expression derived for the magnetic field becomes arbitrarily large as $v \rightarrow a$ and the radiation remains finite and nonzero, whereas it should vanish because of the symmetry of the problem. The part of the electric field that contributes to the total power lies wholly in the magnetic field direction and the direction of the Poynting vector is such that it cannot represent the radiation from a transverse wave. In the normal Cerenkov effect, the wave is transverse in character and the associated electric field is not restricted to the direction of the external magnetic field.

Because the ionospheric plasma behaves much like a conical waveguide, radiation of much lower intensity is produced even when the particle velocity is less than the phase velocity. This latter effect does not arise when the particle traverses an isotropic medium. The radiation is contained within the surface of a cone of very small angle which under no circumstances lies outside of the Alfven cone of angle $\theta_a = \tan^{-1} \frac{1}{\sqrt{\gamma}}$ Again the expressions for the magnetic field and the total power are incorrect as v approaches a, this time from below. The intensity of this effect is so low as to render it undetectable by conventional means.

The approximations used in arriving at these results were:

a) collisions were neglected

b) plasma considered at zero-temperature

c) ELF approximation--wave frequencies less than the ion cyclotron frequency

d) source localized within the Alfven cone.

CHAPTER VI

FINITE TEMPERATURE

The inclusion of finite temperature leads to further modification of the electric susceptibility tensor which in turn leads to temperature dependent Landau or "thermal" damping of the ELF modes previously discusse It has been shown[3] that the purel·· thermal effects of collisions are important in the F-layer where the temperature of the electron gas, $T(^{o}K)$ is equal to or greater than $2 \times 10^{-12}\ a^{2}$(c.g.s), where $\underline{a}$ is the Alfven or phase velocity for MHD waves propagating in the medium. For the zero-temperature case, the various diffusion coefficients (one for each species) appearing in the Fokker-Planck operator were set equal to zero. With the existence of finite temperatures, the ratio $\frac{D}{\eta}$ appears both in the kinetic Green's function and the ensuing susceptibilities.

THE ELECTRIC SUSCEPTIBILITY TENSOR

In the report of Cantor, Keilson and Schneider,[3] the temperature-dependent conductivity tensor was derived using the Fokker-Planck Equation as a starting point. We refer to that report for details. The result of interest is

$$\sigma = \varepsilon_o \omega_p^{\ 2} \int_o^\infty dT e^{i\omega T} e^{-\mu k \cdot B \cdot k} \left[\frac{dA}{dT} - \mu A \cdot kk \cdot A \right] \tag{6-1}$$

where A and B are matrices with nonzero components (neglecting collision

$$\begin{aligned} A_{zz} &= T \\ A_{xx} = A_{yy} &= \frac{\sin \Omega T}{\Omega} \\ A_{xy} = -A_{yx} &= \frac{1-\cos \Omega T}{\Omega} \\ B_{xx} = B_{yy} &= \frac{1-\cos \Omega T}{\Omega^2} \\ B_{zz} &= \frac{T^2}{2} \end{aligned} \tag{6-2}$$

The conductivity tensor is related to the susceptibility tensor $\hat{X}$ defined by

$$\hat{X} = \frac{i}{\varepsilon_o \omega} \sigma \tag{6-3}$$

which appears explicitly in the matrix equation relating the electric field to the external current(Equation (2-27)).

Then

$$\hat{X} = i \frac{\omega_p^2}{\omega} \int_o^\infty dT e^{i\omega T} e^{-\mu k \cdot B \cdot k} \left[\frac{dA}{dT} - \mu A \cdot kk \cdot A\right] \tag{6-4}$$

$\frac{dA}{dT}$ and $A \cdot kk \cdot A$, which appear in the integrand, are both matrices.

$$\frac{dA}{dT} = \begin{pmatrix} \cos \Omega T & \sin \Omega T & 0 \\ -\sin \Omega T & \cos \Omega T & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6-5}$$

$$A \cdot kk \cdot A = Q \tag{6-6}$$

$$Q_{xx} = \frac{\sin^2 \Omega T}{\Omega^2} k_x^2 - \frac{(1-\cos \Omega T)^2}{\Omega^2} k_y^2 \tag{6-7}$$

$$Q_{yy} = \frac{\sin^2 \Omega T}{\Omega^2} k_y^2 - \frac{(1-\cos \Omega T)^2}{\Omega^2} k_x^2 \tag{6-8}$$

$$Q_{zz} = T^2 k_z{}^2 \tag{6-9}$$

$$Q_{xy} = k_T^2 \frac{\sin\Omega T(1-\cos\Omega T)}{\Omega^2} + \frac{k_x k_y}{\Omega^2} \left[\sin^2 \Omega T + (1-\cos\Omega T)\right]^2 \tag{6-10}$$

$$Q_{yx} = \frac{k_T^2 \sin\Omega T(1-\cos\Omega T)}{\Omega^2} - \frac{k_x k_y}{\Omega^2} \left[\sin^2 \Omega T + (1-\cos\Omega T)\right]^2 \tag{6-11}$$

$$Q_{xz} = \frac{k_z T}{\Omega} \left[k_x \sin\Omega T + k_y (1-\cos\Omega T)\right] \tag{6-12}$$

$$Q_{zx} = \frac{k_z T}{\Omega} \quad k_x \sin\Omega T - k_y(1-\cos\Omega T) \tag{6-13}$$

$$Q_{yz} = \frac{k_z T}{\Omega} \quad k_y \sin\Omega T - k_x(1-\cos\Omega T) \tag{6-14}$$

$$Q_{zy} = \frac{k_z T}{\Omega} \quad k_y \sin\Omega T + k_x(1-\cos\Omega T) \tag{6-15}$$

First let us consider the case of zero external magnetic field. In the absence of both η and Ω, the matrices B and A are diagonal and isotropic

$$B = \frac{T^2}{2} \underset{\approx}{1} \; : \; A = T \underset{\approx}{1} \tag{6-16}$$

Then

$$\hat{X} = i \frac{\omega_p^2}{\omega} \int_0^\infty dT \exp(-i\omega T) \exp(-\frac{\mu}{2} k^2 T^2) \left[\underset{\approx}{1} - \mu T^2 \, k \, k \right] \tag{6-17}$$

The next step in evaluating this integral is to go to the "low temperature" limit. The low-temperature limit is dictated by the electron and ion temperatures prevalent in the ionosphere($> 1000^\circ$K) and is consistent with the low frequency limit under which we are laboring. In order to expand exp $(\frac{-\mu}{2} k^2 T^2)$ in a power series and neglect all terms after the second, the inequality

$$\mu k^2 T^2 << 1 \tag{6-18}$$

must hold. But T is a proper time and is inversely proportional to ω, the only frequency occurring in the integrand. Further, $\mu = \frac{D}{\eta}$ is just the square of the thermal velocity, v_{th}.

Therefore, each species must satisfy

$$\lambda > \frac{2\pi}{\omega} v_{th} \tag{6-19}$$

where $\lambda = \frac{2\pi}{k}$ is the wavelength of the ELF oscillation having angular frequency ω.

The thermal velocities for O_2^+ ions and electrons are $1 \cdot 3 \times 10^5$ and 3×10^7 cm/sec, respectively assuming a temperature of $1000^{\circ}K$ in the F-region. Even in the exosphere, with a temperature of possibly $3000^{\circ}K$. the thermal velocity of electrons does not exceed 5×10^7 cm/sec. Therefore, in order for the inequality to be satisfied, the wavelengths must be greater than 10^7 cm, for a wave with frequency $\omega = 30$ sec^{-1}. However, ELF magnetohydrodynamic waves of this frequency, traversing the exosphere at the Alfven velocity of 5×10^8 cm/sec. do indeed have wavelengths of hundreds of kilometers and hence the two approximations of low-temperature and low frequency are consistent. For the low temperature limit actually places a restriction on the wave number, k, which in turn leads to long wavelengths which are consistent with the additional assumption of extra-low frequencies. Let us speak of frequencies below 30 sec^{-1} as lying in the ultra-low frequency or ULF band. Therefore, from Equation (3-38), after expanding the exponential and performing the T-integrals:

$$\hat{X} = \frac{-\omega_p^2}{\omega^2} \underset{\approx}{1} - \frac{\omega_p^2}{\omega^4} \mu \left(\underset{\approx}{1} k^2 + 2 \underset{\sim}{k} \underset{\sim}{k} \right) \tag{6-20}$$

As was mentioned previously, possible modes of radiation are closely related to the normal modes and an investigation of the dispersion relation governing the latter can yield valuable information concerning the former. The radiation-like denominators appearing in the various Green"s Functions already discussed are, when set equal to zero, nothing more than the dispersion relations governing the normal modes.

The matrix equation involving the electric field is again

$$\left[\left(k^2 - \frac{\omega^2}{c^2}\right) \underset{\approx}{1} - \underset{\sim}{k} \underset{\sim}{k} - \frac{\omega^2}{c^2} \hat{X} \right] E = 0 \text{ for the normal modes} \tag{6-21}$$

Define two scalar quantities by

$$S = K^2 - \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2} + \frac{\omega_p^2}{c^2 \omega^2} \mu k^2 \tag{6-22}$$

and

$$U = 1 - \frac{2\omega_p^2}{c^2\omega^2}\mu \tag{6-23}$$

Then, Equation (6-21) can be rewritten in terms of these two scalars

$$(S\underset{\approx}{1} - U\,k\,k)\,E = 0 \tag{6-24}$$

The necessary and sufficient condition for the existence of nontrivial solutions is that the determinant of coefficients vanish. Therefore

$$\det\,(S\underset{\approx}{1} - U\,k\,k\,) = 0 \tag{6-25}$$

and some simple algebra leads to

$$\det\,(S\underset{\approx}{1} - Uk\,k\,) = S^2(S - Uk^2) \tag{6-26}$$

Hence

$$S^2 = 0 \tag{6-27}$$

and

$$S - Uk^2 = 0 \tag{6-28}$$

The double mode is given by

$$S = k^2 + \frac{\omega_p^2}{c^2} + \frac{\omega_p^2}{c^2}\mu\,\frac{k^2}{\omega^2} - \frac{\omega^2}{c^2} = 0 \tag{6-29}$$

or solving for k in terms of ω in the limit of the ULF approximation, $\omega \ll \omega_p$

$$k^2 + \frac{\omega^2}{\mu} = 0 \tag{6-30}$$

which is analogous to the damped mode described by the Yukawa potential which was previously discussed. The third mode is given by

$$S - Uk^2 = 3\,\frac{\omega_p^2}{c^2\omega^2}\,\mu k^2 + \frac{\omega_p^2}{c^2} - \frac{\omega^2}{c^2} = 0 \tag{6-31}$$

and the resultant dispersion relation is

$$k^2 + 3\,\frac{\omega^2}{\mu} = 0 \tag{6-32}$$

which again represents an isotropic damped wave. Therefore, the case of low temperature, in the limit $\mu k^2 << \omega^2$, with the absence of an external magnetic field results in two independent modes both of which are isotropically damped. The damping is frequency dependent and depends on the thermal velocity $v_{th} = \mu^{1/2} = 10^7$ cm/sec.

Returning to the excitation problem, the two Green's Functions for the low-temperature, zero-magnetic field case are

$$H_1(r\omega) = \int \frac{d^3k}{(2\pi)^3}\,\frac{e^{ik\cdot r}}{k^2\left(1+\frac{\omega_p^2}{\omega^2}\frac{\mu}{c^2}\right)+\frac{\omega_p^2}{c^2}}$$

$$= \frac{1}{4\pi|r|}\left(\frac{1}{1+\frac{\omega_p^2}{\omega^2}\frac{\mu}{c^2}}\right)\exp\left(\frac{-\omega_p|r|/c}{\left(1+\frac{\omega_p^2}{\omega^2}\frac{\mu}{c^2}\right)^{1/2}}\right) \tag{6-33}$$

and

$$H_2 = \frac{1}{4\pi|r|}\left(\frac{1}{1+3\frac{\omega_p^2}{\omega^2}}\right)\exp\left(\frac{-\omega_p|r|/c}{\left(1+3\frac{\omega_p^2}{\omega^2}\frac{\mu}{c^2}\right)^{1/2}}\right) \tag{6-34}$$

But

$$\frac{\omega_p^2}{\omega^2}\,\frac{v_{th}^2}{c^2} \sim 10^8 \text{ for } \omega = 10 \text{ sec}^{-1}$$

Therefore

$$H_1 \sim \frac{1}{\frac{\omega_p^2}{\omega^2}\frac{\mu}{c^2}}\;\frac{\exp\left(-\frac{\omega|r|}{v_{th}}\right)}{4\pi|r|} \text{ and } H_2 \sim \frac{1}{\frac{3\omega_p^2}{\omega^2}\frac{\mu}{c^2}}\;\frac{\exp\left(-\frac{\omega|r|}{v_{th}}\right)}{4\pi|r|} \qquad (6\text{-}35)$$

as constrasted to the zero-temperature case

$$G_2 \sim \frac{\exp\left(-\frac{\omega_p|r|}{c}\right)}{4\pi|r|} \qquad (6\text{-}36)$$

The damping in the latter case is much greater than the thermal damping occurring when finite temperatures are included. However, the new Green's Functions have a constant factor of 10^{-8} multiplying them, which is absent from G_2. Thus, the inclusion of low temperature results in a greater damping distance but the amplitude of the initial oscillatic is greatly depressed, and the wave velocity is greatly reduced.

LOW TEMPERATURE,FINITE MAGNETIC FIELD

We now consider the case of a finite, external magnetic field, B_o, whose magnitude is contained in the ion and electron gyrofrequencies $\Omega_{\pm}$.

Most ELF magnetohydrodynamic waves that have been detected at the earth have been observed to propagate along the lines of the earth's general magnetic field. The magnetic field direction, has a low-frequency conductivity much greater than that in any other direction and serves almost as a waveguide. For every possible ELF wave heretofore discussed, excluding the isotropic mode, the propagation is in the magnetic field direction. Since one-dimensional propagation along the field lines seems to be the recurrent and dominant physical phenomenon, we shall limit the ensuing discussion to this particular case of interest by imposing the condition

$$k = k_z \hat{z} \tag{6-37}$$

on the propagation vector.

This greatly simplifies the calculations for the matrix Q = A.kkA., now has only one nonvanishing component,

$$Q_{zz} = T^2 k_z^{\ 2} \tag{6-38}$$

as can be seen directly from Equations (6-5)-(6-15). The nonvanishing components of the special susceptibility tensor, X, are then given by

$$\hat{X}_{zz} = i \frac{\omega_p^2}{\omega} \int_0^\infty dTe^{i\omega T} e^{-\mu k_z^2 \frac{T^2}{2}} (1-\mu k_z^{\ 2}T^2) \tag{6-39}$$

$$\hat{X}_{xx} = \hat{X}_{yy} = \frac{i\omega_p^{\ 2}}{\omega} \int_0^\infty dTe^{i\omega T} e^{-\mu k_z^2 \frac{T^2}{2}} \cos \Omega T \tag{6-40}$$

$$\hat{X}_{xy} = - \hat{X}_{yx} = \frac{i\omega_p^{\ 2}}{\omega} \int_0^\infty dTe^{i\omega T} e^{-\mu k_z^2 \frac{T^2}{2}} \sin \Omega T \tag{6-41}$$

It will be shown shortly that each of the above integrals can be explicitly written in terms of its real and imaginary parts, and that in each case, one of these parts can be evaluated exactly whereas the low-temerature and/or low frequency approximations discussed above must be invoked to effect the other solution.

For example, the z-z component may be rewritten:

$$\hat{X}_{zz} = \frac{\omega_p^2}{\omega} \int_0^\infty dTe^{-\mu k_z^2 \frac{T^2}{2}} (-\sin \omega T + i \cos \omega T)(1-\mu k_z^{\ 2}T^2)$$

$$= \frac{\omega_p^2}{\omega}\left(1 + \mu k_z^{\,2} \frac{\partial^2}{\partial\omega^2}\right) \int_0^\infty dT e^{-\mu k_z^{\,2} \frac{T^2}{2}} (-\sin \omega T + i \cos \omega T) \tag{6-42}$$

The imaginary part of this expression is easily evaluated

$$\int_0^\infty \cos \omega T e^{-\mu k_z^{\,2} \frac{T^2}{2}} dT = \frac{1}{4}\left[\int_\infty^\infty dT e^{-\mu k_z^{\,2}\ T\ 2/2} (e^{i\omega T} + e^{-i\omega T})\right.$$

$$= \exp\left(-\frac{\omega^2}{2\mu k_z}\right)\left(\frac{2\pi}{\mu k_z^{\,2}}\right)^{1/2} \tag{6-43}$$

Therefore from Equation (6-42)

$$X_{zz} = -\left(1 + \frac{\mu k_z^{\,2} \partial^2}{\partial\omega^2}\right)\omega_p^2 \int_0^\infty \sin \omega T\ e^{-\mu k_z^{\,2} T2/2} dT + i\omega_p^2\sqrt{\frac{\pi}{2}}$$

$$(\mu k_z^{\,2})^{-3/2} \exp\left(-\frac{\omega^2}{2\mu k_z^{\,2}}\right) \tag{6-44}$$

To evaluate the real part, we go to the low-temperature approximation

$$\mu k_z^{\,2} T^2 << 1$$

which puts a restriction on the wavelength $\lambda > \frac{2\pi v_{th}}{\omega}$ (for each species), as discussed previously. The exponential can then be expanded in a power series

$$\text{Re } X_{zz} = \omega_p^2\left(1 + \mu k_z^{\,2} \frac{\partial^2}{\partial\omega^2}\right) \int_0^\infty dT \sin \omega T\left(1 - \mu \frac{k_z^{\,2} T^2}{2}\right)$$

$$= \omega_p^2\left(1 + \mu k_z \frac{\partial^2}{\partial\omega^2}\right)\left(\frac{1}{\omega} + \frac{\mu k_z}{\omega^3}\right) \approx -\frac{\omega_p^2}{\omega^2} \tag{6-45}$$

This is precisely the zero temperature result for X_{zz}. Thus, the presence of a finite temperature serves to make X_{zz} a complex quantity whose imaginary part is wave-number dependent. The evaluation of $\hat{X}_{xx}$ and $\hat{X}_{zz}$ follow similar lines

$$\hat{X}_{xx} = \frac{\omega_p^2}{2\omega} \int_0^\infty dTe^{-\mu k_z^2 T^2} \Big[\sin(\Omega-\omega)T - \sin(\Omega+\omega)T$$

$$+ i(\cos(\Omega-\omega)T + \cos(\Omega+\omega)T)\Big] \qquad (6\text{-}46)$$

Again the imaginary part can be determined exactly. Making use of our previous result

$$\int_0^\infty \cos aTe^{-\frac{b}{2}T^2} = \sqrt{\frac{\pi}{2}}\, b^{-1/2} \exp\left(-\frac{a^2}{2b}\right) \qquad (6\text{-}47)$$

$$\mathrm{Im}\hat{X}_{xx} = \frac{\omega_p^2}{\omega}\sqrt{\frac{\pi}{8}}\,(\mu k_z^2)^{-/2} \left[\exp\left(-\frac{(\Omega-\omega)^2}{2\mu k_z^2}\right) + \exp\left(-\frac{(\Omega+\omega)^2}{2\mu k_z^2}\right)\right]$$

$$(6\text{-}48)$$

In the limit of extra low frequencies, $\omega << \Omega < \omega_p$. This expression reduces to

$$\mathrm{Im}\hat{X}_{xx}(\mathrm{ELF}) = \frac{\omega_p^2}{\omega}\sqrt{\frac{\pi}{2}}\,(\mu k_z^2)^{-1/2} \exp\left[-\frac{\Omega^2}{2\mu k_z^2}\right] \qquad (6\text{-}49)$$

To evaluate the real part, we make use of the low frequency approximation to expand sin $(\Omega-\omega)T$ and sin $(\Omega+\omega)T$ in powers of ωT about $\omega = 0$.

$$\sin(\Omega+\omega)T = \sin\Omega T + \omega T \cos\Omega T + \ldots \qquad (6\text{-}50)$$

$$\sin(\Omega-\omega)T = \sin\Omega T - \omega T \cos\Omega T + \ldots \qquad (6\text{-}51)$$

Then, from Equation (6-46)

$$\text{Real } \hat{X}_{xx} = -\omega_p^2 \int_0^\infty dTe^{-\mu k_z{}^2 T^2/2} \; T \cos \Omega T$$

$$= + \frac{\omega_p^2}{\mu k_z{}^2} \int_0^\infty dT \frac{d}{dT} \left(e^{-\mu k_z{}^2 T^2/2} \right) \cos \Omega T \tag{6-52}$$

and integrating by parts

$$= \frac{\omega_p^2}{\mu k_z{}^2} \left[\Omega \int_0^\infty dT \sin\Omega T e^{-\mu k_z{}^2 T^2/2} -1 \right] \tag{6-53}$$

Again, going to the low temperature limit, we assume $\mu k_z{}^2 << \Omega^2$ which is again compatible with our low-frequency assumption in that it puts a less stringent condition on the wavelength, namely $\lambda > \frac{2\pi v_{th}}{\Omega}$ or $\lambda > 10^4$cm (for ions) which is certainly satisfied by any extra-low frequency magnetohydrodynamic disturbance propagating at the Alfven velocity. Then

$$\text{Real } \hat{X}_{xx} = \frac{\omega_p^2}{\mu k_z{}^2} \left[\Omega \int_0^\infty dT \sin\Omega T (1-\mu k_z{}^2 T^2/2) -1 \right]$$

$$= \frac{\omega_p^2}{\Omega^2} \tag{6-54}$$

The real part of X_{xy} can be determined exactly. From Equation (6-41)

$$\text{Re } \hat{X}_{xy} = -\frac{\omega_p^2}{2\omega} \int_0^\infty dTe^{-\mu k_z{}^2 T^2/2} \left[\cos(\Omega-\omega)T - \cos(\Omega+\omega)T \right]$$

$$= \frac{\omega_p^2}{\omega} \sqrt{\frac{\pi}{8}} (\mu k_z{}^2)^{-1/2} \left(\exp\left[-\frac{(\Omega+\omega)^2}{2\mu k_z{}^2} \right] - \exp\left[-\frac{(\Omega-\omega)^2}{2\mu k_z{}^2} \right] \right) \tag{6-55}$$

and in the ELF approximation

$$\mathrm{Re}\ \hat{X}_{xy\,(ELF)} = + \frac{\omega_p^2\Omega}{(\mu k_z^2)^{3/2}}\sqrt{\frac{\pi}{2}}\ \exp\left(-\frac{\Omega^2}{2\mu k_z^2}\right) \tag{6-56}$$

$$\mathrm{Im}\ \hat{X}_{xy} = \frac{\omega_p^2}{2\omega}\int_0^\infty dT\ \exp(-\mu k_z^2 T^2/2)\quad \sin(\Omega+\omega)T + \sin(\Omega-\omega)T \tag{6-57}$$

Again we expand $\sin(\Omega-\omega)T$ and $\sin(\Omega+\omega)T$

$$\mathrm{Im}\ \hat{X}_{xy} = \frac{\omega_p^2}{\omega}\int_0^\infty dT\ \exp(-\mu k_z^2 T^2/2)\ \sin\Omega T$$

In the limit of low temperature and low frequency the inequality $\mu k_z^2 << \Omega^2$ is satisfied and hence

$$\mathrm{Im}\ \hat{X}_{xy} = \frac{\omega_p^2}{\omega}\int_0^\infty dT\ \sin\Omega T(1-\mu k_z^2 T^2/2)$$

$$= \frac{\omega_p^2}{\Omega\omega} + \frac{\mu k_z^2\omega_p^2}{\Omega^3\omega} \tag{6-58}$$

The first term vanishes when summed over the two species because of the condition of charge neutrality in the plasma. Therefore

$$\mathrm{Im}\ \hat{X}_{xy} = \frac{\mu k_z^2\omega_p^2}{\Omega^3\omega} \tag{6-59}$$

Thus, in the low-temperature, low-frequency limit, the electric susceptibility tensor is given by

$$\hat{X}_{zz} = -\frac{\omega_p^2}{\omega^2} + i\,\frac{\omega_p^2\omega}{(\mu k_z^2)^{3/2}}\sqrt{\frac{\pi}{2}}\ \exp\left(-\frac{\omega^2}{2\mu k_z^2}\right) \tag{6-60}$$

$$\hat{X}_{xx} = \hat{X}_{yy} = \frac{\omega_p^2}{\Omega^2} + i \frac{\omega_p^2}{\omega} \sqrt{\frac{\pi}{2}} (\mu k_z^2)^{-1/2} \exp\left(-\frac{\Omega^2}{2\mu k_z^2}\right) \tag{6-61}$$

$$\hat{X}_{xy} = -\hat{X}_{yx} = \frac{\omega_p^2 \Omega}{(\mu k_z^2)^{3/2}} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{\Omega^2}{2\mu k_z^2}\right) + i \frac{\mu k_z^2 \omega_p^2}{\Omega^3 \omega} \tag{6-62}$$

It should be noted that the exponential term in each of these three expressions is a very small quantity because of the low-temperature approximation

$$\mu k_z^2 << \omega^2, \Omega^2.$$

In the zero-temperature case, in the absence of collisions, $\hat{X}_{xx}$ and $\hat{X}_{zz}$ were both real and wave-number independent, and $\hat{X}_{xy}$ vanished. From the structure of the above expressions, one can easily see that the dependence of $\hat{X}(k\omega)$ on k is entirely contained in the temperature dependence as was previously stated.

Again, we shall first obtain the dispersion relations from the normal modes and then construct the temperature-dependent Green's Functions.

The normal mode equation is again

$$\left[\left(k_z^2 - \frac{\omega^2}{c^2}\right) \underset{\approx}{1} - k_z k_z - \frac{\omega^2}{c^2} \hat{X}\right] \underset{\approx}{E} = 0$$

The vanishing of the determinant of the coefficients leads to

$$\frac{\omega^2}{c^2}(1+X_{zz}) \left[\left(k_z^2 - \frac{\omega^2}{c^2} (1+X_{xx})\right)^2 + \left(\frac{\omega^2}{c^2} X_{xy}\right)^2\right] = 0 \tag{6-63}$$

The resultant modes are then given by the dispersion relations

$$(1+\hat{X}_{zz}) = 0 \tag{6-64}$$

$$k_z^2 - \frac{\omega^2}{c^2}(1+\hat{X}_{xx} \pm i\,\hat{X}_{xy}) = 0 \tag{6-65}$$

The first of these equations represents a longitudinal oscillation similar to the plasma oscillation occurring in the zero-temperature plasma.

We shall restrict our attention to the two modes given by Equation (6-65) and attempt to determine the criteria for thermal damping

$$\hat{X}_{xx} \pm i\,\hat{X}_{xy} = \frac{\omega_p^2}{\Omega^2} \mp \frac{\mu k_z{}^2\omega_p^2}{\Omega^3\omega}$$

$$+ i\sqrt{\frac{\pi}{2}}\ \exp\left(-\frac{\Omega^2}{2\mu k_z{}^2}\right)\frac{\omega_p^2\Omega}{(\mu k_z{}^2)^{3/2}}\left[\frac{\mu k_z{}^2}{\omega\Omega} \pm 1\right]$$

In the limit of low temperatures and low frequencies

$$\mu k_z{}^2 << \omega^2 << \Omega^2$$

Therefore

$$\hat{X}_{xx} \pm i\,\hat{X}_{xy} = \frac{\omega_p^2}{\Omega^2} \mp \frac{\mu k_z{}^2\omega_p^2}{\Omega^3\omega} \pm i\sqrt{\frac{\pi}{2}}\ \exp\left(-\frac{\Omega^2}{2\mu k_z{}^2}\right)\frac{\omega_p^2\Omega}{(\mu k_z{}^2)^{3/2}} \tag{6-66}$$

This expression is actually a sum over the two species, but the values of μ, ω_p and Ω in both the F-region and the exosphere are such that the ion terms predominate. Typical values of these parameters in both regions are listed below in Table 1.

	F-Region	Exosphere
ω_{p-}	1.2×10^8 sec^{-1}	1.0×10^5 sec^{-1}
ω_{p+}	7.0×10^5 sec^{-1}	7.0×10^2 sec^{-1}
$\Omega-$	1.0×10^7 sec^{-1}	1.0×10^5 sec^{-1}
$\Omega+$	3.0×10^2 sec^{-1}	3.0 sec^{-1}
$\mu-$	2.5×10^{15} cm^2/sec^2	5.0×10^{15} cm^2/sec^2
$\mu+$	1.0×10^{11} cm^2/sec^2	1.0×10^{11} cm^2/sec^2
a (Alfven velocity	1.0×10^7 cm/sec	1.0×10^8 cm/sec
T(kinetic temp)	1000°K	3000°K

Henceforth, the quantities ω_p, Ω, and μ appearing in the dispersion relation shall be the ionic contributions only.

$$k_z^2 - \frac{\omega^2}{c^2}(1+ X_{xx} \pm i X_{xy}) =$$

$$k_z^2 - \frac{\omega^2}{c^2} - \frac{\omega^2}{c^2}\frac{\omega_p^2}{\Omega^2} \pm \frac{\mu k_z{}^2\omega_p^2}{\Omega^3 c^2} \mp i\sqrt{\frac{\pi}{2}}\; \frac{\omega_p^2\omega^2\Omega}{c^2(\mu k_z{}^2)^{3/2}} \; \exp\left(-\frac{\Omega^2}{2\mu k_z{}^2}\right) \tag{6-67}$$

If we neglect the imaginary part of Equation (6-68) then to a first approximation

$$k_z^2 \approx \frac{\omega^2/a^2}{1 \pm \frac{\mu}{a^2}\frac{\omega}{\Omega}} \tag{6-68}$$

Since in the ELF limit $\omega << \Omega$, the denominator can be expanded to obtain

$$k_z^2 \approx \frac{\omega^2}{a^2}\left(1 \mp \frac{\mu}{a^2}\frac{\omega}{\Omega}\right) \tag{6-69}$$

As $\frac{\mu}{a^2}\frac{\omega}{\Omega}$ is of the order of 10^{-4}, k_z^2 is nearly equal to $\frac{\omega^2}{a^2}$ in the first approximation. One can therefore obtain the next higher approximation by replacing k_z^2 everywhere in the imaginary part of Equation (6-70) by $\frac{\omega^2}{a^2}$. Therefore

$$k_z^2 \approx \frac{\omega^2}{a^2} \mp i\sqrt{\frac{\pi}{2}}\,\frac{\omega^2}{c^2}\,\frac{\Omega\omega_p^2}{\left(\mu\frac{\omega^2}{a^2}\right)^{3/2}}\,\exp\left(-\frac{\Omega^2 a^2}{2\mu\omega^2}\right) \tag{6-71}$$

and

$$k_z \approx \frac{\omega}{a}\left[1 \mp i\sqrt{\frac{\pi}{8}}\,\frac{a^5}{c^2\mu^{3/2}}\,\frac{\Omega\omega_p^2}{\omega^3}\,\exp\left(-\frac{\Omega^2 a^2}{2\mu\omega^2}\right)\right] \tag{6-72}$$

It can easily be seen from this last equation that for $\omega < 10^4\ \text{sec}^{-1}$ the exponential dominates the damping term and tends toward zero. Thus for any frequencies in the ultra-low frequency (ULF) range for which the low temperature and low frequency approximations are satisfied, the damping distance becomes very great, and the low-temperature thermal damping is negligible. Only for angular frequencies of 10^4 and greater does the imaginary part of the wave vector, as given by Equation (6-72) become the order of the real part. Hence, the effects of low-temperature thermal damping may be neglected for ultra-low frequency one-dimensional propagation along the magnetic field lines.

The other mode contained in Equation (6-63) is analogous to the longitudinal plasma oscillation resulting in the zero temperature case $1 - \frac{\omega_p^2}{\omega^2} = 0$.

Now, in the low-temperature limit, $\hat{X}_{zz}$ is wave-number dependent, and the plasma oscillation is replaced by

$$1 - \frac{\omega_p^2}{\omega^2} + i \frac{\omega_p^2 \omega}{(\mu k_z^2)^{3/2}} \sqrt{\frac{\pi}{2}} \exp\left(- \frac{\omega^2}{2\mu k_z^2}\right) = 0 \tag{6-73}$$

$$- \frac{\omega_p^2}{\omega^2} \left(1 - i \sqrt{\frac{\pi}{2}}\, x^3 \exp\left(- \frac{x^2}{2}\right)\right) = 0 \tag{6-74}$$

with

$$x^2 = \frac{\omega^2}{\mu k_z^2} \tag{6-75}$$

Then,

$$x^3 e^{-\frac{x^2}{2}} = -\sqrt{\frac{2}{\pi}}\, i \tag{6-76}$$

Assume a solution of the form $x = Ai$

$$-A^3 i\, e^{+A^{2/2}} = - i \sqrt{\frac{2}{\pi}}$$

Therefore, solving for A by trial and error, we find

$$A = .95$$

Thus

$$x = \frac{\omega}{v_{th} k_z} = .95\, i$$

$$k_z = 1.05 \frac{\omega}{v_{th}}\, i \tag{6-77}$$

This longitudinal mode represents a damped oscillation with a damping distance of 100 meters, travelling at the ion thermal velocity. Thus for the case of one-dimensional propagation along the field lines, the zero-temperature Alfven mode $k_z = \frac{\omega}{a}$ is unchanged by the inclusion of temperature in the ULF limit, whereas the longitudinal plasma oscillation transcends to a damped oscillation. Again, the most important feature of the above discussion is that the one-dimensional magnetohydrodynamic wave propagating at the Alfven velocity is unaffected, in the ULF limit, by the presence of a finite temperature.

CHAPTER VII

A MANY-BODY DERIVATION OF THE BOLTZMANN EQUATION FOR HARD-SPHERE MOLECULES

The distribution functions f^1, f^2, $f^3 \cdots$, are connected by a chain of equations which describe in an obvious way the motion of one or two or three "typical particles" in the system relative to the rest of the system. The first two equations in this chain have the form

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla + a \cdot \nabla_v\right) f(rvt) + \frac{1}{m} \int_{r'} \int_{v'} F(r-r') \cdot \nabla_v f^2(rvr'v't) = 0 \tag{7-1}$$

and

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla + v' \cdot \nabla' + a \cdot \nabla_v + a' \cdot \nabla_{v'} + \frac{1}{m} F(r-r') \cdot \nabla_v + \frac{1}{m} F(r'-r) \cdot \nabla_v\right.$$

$$\times f^2(rvr'v't) + \frac{1}{m} \int_{r''} \int_{v''} (F(r-r'') \cdot \nabla_v + F(r'-r'') \cdot \nabla_{v'})$$

$$\times f^3(rvr'v'r''v''t) = 0 \tag{7-2}$$

The operator on f in Equation (7-1) describes the motion of a single particle under the influence only of an external force. The operator on f^2 in Equation (7-2) describes the motion of two particles moving under the combined influence of the external force and the interparticle force. The integrated force terms are due to the combined average influence of more or less "near" particles, and is less significant at low densities than high. The reason that a knowledge of a high degree of correlation is needed in these terms is that the distribution of "near" particles is not unrelated to the same force which is being taken account of.

Our intuition tells us that a large class of problems ought to be solvable using only Equation (7-1) and some simple assumption about f^2. Likewise another large class of problems, very likely encompassing the first class, ought to require no more than Equations (7-1) and (7-2) and some simple assumption about f^3. Indeed the "self-consistent force" approximation, and the "Boltzmann collision hypothesis" are the direct expression of this intuition. We will proceed on the basis of a slightly different philosophy which has, in practice, no more or less valid basis than the aforementioned ideas, but has certain obvious mathematical extensions. This is the idea of successive noncorrelations.

When no <u>internal forces</u> act (we use a subscript zero to indicate this case), there can be no correlations among the particle motions. Either by inspection or intuitively it follows, for example, that

$$f_o^2(rv,r'v't) = f_o(rvt) \times f_o(r'v't) \tag{7-3}$$

What information does the nontrivial function

$$f_{N.C.}^2(rv,r'v't) = f(rvt)f(r'v't) \tag{7-4}$$

imply? Let us find its equation, and compare it with Equation (7-2).

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla + v' \cdot \nabla' + a \cdot \nabla_v + a' \cdot \nabla_{v'}\right) f_{NC}^2$$

$$+ \frac{1}{m} \int_{r''} \int_{v''} \Big[F(r-r'') \cdot \nabla_v \, f(r'v't) f^2(rv,r''v''t)$$

$$+ F(r'-r'') \cdot \nabla_{v'} \, f(rvt) f^2(r'v'r''v''t)\Big] = 0 \tag{7-5}$$

We see that f^2_{NC} is approximately equal to f^2 when the two particles are far enough apart that their mutual force may be neglected, and when the system is such that in the f^3 term only the particles connected by a force need be treated as correlated; that is,

$$f^3(rv,r'v',r''v''t) \Longrightarrow f^2(r'v',r''v''t)\,f(rvt) \tag{7-6}$$

in a term in which r' and r'' are connected by a force. Acting on the belief that in a multiparticle system with two-body forces the two-particle function f^2 will be of greater numerical importance than f^3, no matter how strong the forces are, we conclude that the approximation expressed by (7-6) will be of extreme generality, and especially even so when the arguments r, r' of f^2 are close to one another. But f^2_{NC} is then a bad approximation to f^2. Consequently, Equations (7-1) and (7-2) together with assumption (7-6) in the force term of (7-2) provide a reasonable way to terminate the otherwise endless set of coupled equations.

We can now derive an equation for the quantity

$$D(rv,r'v't) = f^2(rv\ r'v't) - f(rvt)\,f(r'v't) \tag{7-7}$$

Let us first introduce a simplifying notation. Let us number the particles 1, 2, 3, etc., and let the number stand for the full set of position-velocity variables. For example, $1 = (r_1v_1)$, $2 = (r_2v_2)$ $3 = (r_3v_3)\cdots$. The variable t is the same in all functions and may be ignored. The operator

$$L = \frac{\partial}{\partial t} + v_1 \cdot \nabla_1 + a_1 \cdot \nabla_{v'} + v_2 \cdot \nabla_2 + a_2 \cdot \nabla_{v_2} + \cdots \tag{7-8}$$

may be called $L(12\cdots)$. It describes the linear, noncorrelated part of the motion of the "typical particles" 1, 2,$\cdots$. The operator $M(12\cdots)$ will describe the mutual forces. As examples, $M(1) = 0$,

$$M(12) = \frac{1}{m} F(r_1-r_2) \cdot \nabla_{v_1} + \frac{1}{m} F(r_2-r_1) \cdot \nabla_{v_2}$$

$$M(123) = \frac{1}{m}(F(r_1-r_2) + F(r_1-r_3)) \cdot \nabla_{v_1} + \frac{1}{m} (\ ) \cdot \nabla_{v_2} + \frac{1}{m} (\ ) \cdot \nabla_{v_3} \tag{7-9}$$

with the variable in the force terms appearing in cyclic order.

The superscripts on f^1, f^2 ... may be left off where the arguments of the functions are given, for then the "order" of the distribution function is obvious. Finally, we may let

$$\frac{1}{m} F(r_1-r_2) \cdot \nabla_{v_1} = A(1-2) \tag{7-10}$$

so that, for example,

$$M(123) = A(12) + A(13) + A(21) + A(23) + A(31) + A(32) \tag{7-11}$$

In this notation, the two equations we intend to solve are

$$L(1) f(1) + \int_2 A(12) f(12) = 0 \tag{7-12}$$

and

$$[L(12) + M(12)] f(12) + \int_3 (A(13) f(13) f(2) + A(23) f(23) f(1)) = 0 \tag{7-13}$$

Let

$$D(12) = f(12) - f(1) f(2) \tag{7-14}$$

Since

$$L(12)(f(1)f(2)) + \int_3 (A(13)f(13)f(2) + A(23)f(23)f(1)) = 0 \qquad (7\text{-}15)$$

we find

$$\left[L(12) + M(12)\right] D(12) = -M(12)(f(1)f(2)) \qquad (7\text{-}16)$$

The function D(12) may be introduced into the equation for f(1) instead of f(12). Before we do so, though, we can increase the symmetry of our equations by replacing A(12) by M(12) in Equation (7-12). Note that

$$M(12) = A(12) + A(21) \qquad (7\text{-}17)$$

The A(21) term gives zero under the integral $\int_2$, for this contains $\int_{v'} F(r'-r) \cdot \nabla_{v'} f^2(vv')$ which integrates exactly, leaving $f^2(vv')$ evaluated at the limits of the v' integrals. In any physical system there will be no particles of infinite velocity, so that the integrated term is zero. Thus, Equation (7-12) can be rewritten:

$$L(1)f(1) + \int_2 M(12)f(12) = 0 \qquad (7\text{-}18)$$

and the substitution of (7-14) gives

$$\left[L(1) + \int_2 M(12)f(2)\right] f(1) + \int_2 M(12)D(12) = 0 \qquad (7\text{-}19)$$

The operator added to L(1) arises (of course) in the noncorrelation approximation. Written out it is

$$\int_2 M(12)f(2) = \int_2 A(12)f(2) = \frac{1}{m}\int_{r'}\int_{v'} F(r-r')f(r'v't)\cdot\nabla_v$$

It is often referred to as the "average-force" term, and was the basis of Vlasov's and Landau's treatment of the long-range forces in a plasma. As we mentioned f^2_{NC} is (together with (7-6)) a good approximation to f^2 when the particles are "far enough apart." Since the Coulomb forces do not have a finite range, it is difficult to define for them the precise region of validity of the noncorrelation approximation. It proves to be applicable to the plasma, nevertheless, because of the plasma's ability to shield charges. The <u>effective forces</u> in the plasma are of finite range, so that <u>the solutions using the noncorrelative approximation are consistent with the approximation</u>. But when the forces are solely or primarily short-range, this term has contributions only from one or two neighboring particles: in a distribution theory such a constribution is of negligable weight. The last term of Equation (7-19) then is most important.

Our main problem now is to solve (7-16) for D. The equation is inhomogeneous, and may be solved by a Green's function for the particular solution. The homogeneous solution will have an arbitrary constant multiplying it which must be adjusted to the physical problem ("the boundary conditions"). In fact, there cannot be an homogeneous solution -- the constant is zero -- because only that part of the force on a particle $[f(1)]$ must be included which is specifically due to the self-consistently positioned neighboring particles that are within the range of the force. An homogeneous term D^o would make no reference to this self-consistency, but the inhomogeneous solution of Equation (7-16) would, by virtue of the source term on the right hand side.

For the construction of the Green's function we shall need to indicate the time variable t at which the particle variables are specified. Let $G(12t, 34t_o)$ be the solution of

$$(L(12) + M(12))G(12t,34t_o) = \delta(t-t_o)\delta(1-3)\delta(2-4) \tag{7-20}$$

In the usual pictorial manner of speaking, G is said to take particles 3 and 4 at time t_o into particles 1 and 2 at time t. That is, G describes the details of the orbit and momentum exchange of two particles under the influence of the inter-particle force. In full detail, but leaving out the external acceleration A, we have

$$\left(\frac{\partial}{\partial t} + v_1 \cdot \nabla_1 + v_2 \cdot \nabla_2 + \frac{1}{m} F(r_1-r_2) \cdot \nabla_{v_1} + \frac{1}{m} F(r_2-r_1) \cdot \nabla_{v_2}\right)$$

$$\times\, G(r_1v_1r_2v_2t;\ r_3v_3r_4v_4t_o) = \delta(t-t_o)\delta^3(r_1-r_3)\delta^3(v_1-v_3)\delta^3(r_2-r_4)$$

$$\times\, \delta^3(v_2-v_4) \tag{7-21}$$

All of the delta-functions, with the exception of the first, are three-dimensional, e.g.,

$$\delta^3(r_1-r_3) = \delta(x_1-x_3)\delta(y_1-y_3)\delta(z_1-z_3) \tag{7-22}$$

We write the solution as

$$D(12t) = -\int_3\int_4\int_{t_o} G(12t,34t_o)M(34)f(3t_o)f(4t_o)) \tag{7-23}$$

Application of L(12) + M(12) to this equation leads to Equation (7-16) in view of Equation (7-20).

There remains one element of uncertainty in G -- it may be "retarded" or "advanced" or a mixture of the two. Looking at Equation (7-23) we realize that D(12t) should depend only on the earlier particle distributions, since all of the dynamical processes are causal. This means that $G(t-t_o)$ should be "retarded", i.e., $G = 0$ if $t < t_o$.

Our interest in the solution of the two coupled equations stems, in the present instance, from a desire to treat collisions more accurately than is possible with the Fokker-Planck operator. We can, for this purpose, try to solve Equation (7-21) without the external acceleration a, which is almost never as strong as the internal accelerations $\frac{1}{m} F(r-r')$, and acts mainly to provide a net drift superimposed on the "chaotic" collisional-diffusion processes. Furthermore, the quantity D gives the "closer-in" effects when contrasted with the "long-range" effects of f^2_{NC}. Hence, as a kind of orientation in the problem of collisions, and also for an improved model for collisions in the plasma, we treat the problem of hard spheres colliding. This is certainly an improvement over the Fokker-Planck procedure which applies, as we have noted before, to the motion of large molecules suffering many small impacts in the surrounding medium of small molecules. Whether the collisions are among electrons, among ions, between ions and electrons, or between charged-particles and neutral molecules, we expect some improvement in description.

Let us carry out the analysis as far as possible without specializing the force. Hard-sphere collisions will prove to simplify the final analysis. Since we know that in a two-particle collision total momentum is concerned, and since the force lies along the line between the centers of the (pt.) molecules, we shall introduce appropriate variables. It is enough, for the present, to treat one kind of molecule to illustrate our ideas.

The center of mass (R), the relative distance (r), the center of momentum, V, and the relative velocity (v) of particles 1 and 2 are

$$R = \frac{1}{2} (r_1 + r_2) \qquad V = \frac{v_1 + v_2}{2}$$

$$r = r_1 - r_2 \qquad v = (v_1 - v_2)$$

$$r_1 = R + \frac{1}{2} r, \; r_2 = R - \frac{1}{2} r, \; v_1 = V + \frac{1}{2} v, \; v_2 = V - \frac{1}{2} v \tag{7-24}$$

A similar set of variables, constructed from particles 3 and 4 will be called $R_o V_o r_o v_o$. It is easy to verify that the gradients transform as follows

$$\nabla_{r_1} = \frac{1}{2} \nabla_R + \nabla_r \qquad \nabla_{v_1} = \frac{1}{2} \nabla_V + \nabla_v$$

$$\nabla_{r_2} = \frac{1}{2} \nabla_R - \nabla_r \qquad \nabla_{v_2} = \frac{1}{2} \nabla_V - \nabla_v \tag{7-25}$$

so that

$$v_1 \cdot \nabla_{r_1} + v_2 \cdot \nabla_{r_2} = v \cdot \nabla_r + V \cdot \nabla_R \tag{7-26}$$

and

$$\nabla_{v_1} - \nabla_{v_2} = 2\nabla_v \tag{7-27}$$

Furthermore, since the Jacobian of the transformation (7-24) is unity, the delta-function term in Equation (7-21) may be rewritten directly in terms of the new variables. The resulting equation makes use of Newton's third law:

$$F(r_2 - r_1) = -F(r_1 - r_2) \tag{7-28}$$

and becomes

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla_R + v \cdot \nabla_r + \frac{2}{m} F(r) \cdot \nabla_v\right) G(RVrvt, R_o V_o r_o v_o t_o)$$

$$= \delta(t-t_o)\ \delta(R-R_o)\ \delta(V-V_o)\ \delta(r-r_o)\ \delta(v-v_o) \tag{7-29}$$

We have used the same symbol for the Green's function written in terms of the new variables. We shall speak quite generally of Equation (7-29) as describing a collision, although it will describe a bound system also when the forces can support one. Inspection of Equation (7-20) shows directly that the total "momentum" V is unchanged during the collision. Hence G contains $\delta(V-V_o)$ as a factor. Further, we can also remove a factor expressing the fact that the center of mass moves in a straight line:

$$G = \delta(V-V_o)\delta(R-R_o - V_o(t-t_o))G(rvt,r_ov_ot_o) \tag{7-30}$$

where

$$\left[\frac{\partial}{\partial t} + v \cdot \nabla_r + \frac{2}{m} F(r) \cdot \nabla_v\right] G(rv,r_ov_o)$$

$$= \delta(r-r_o)\delta(v-v_o)\delta(t-t_o) \tag{7-31}$$

For a force which can be derived from a central potential, X:

$$F(r) = - \nabla X(|r|) = - \frac{\partial}{\partial |r|} X \cdot \nabla |r| = - X' \hat{r} \tag{7-32}$$

in which $\hat{r}$ is the unit vector along the radial direction (from the origin of coordinates). Consequently, a spherical coordinate system for r together with a suitable chosen velocity coordinate system <u>at each point of space</u> will simplify Equation (7-31) in such a way as to make clear the fact that the force between the two particles only causes a radial acceleration. The transformations for the coordinates are

$$\begin{aligned} x &= |r| \sin \Theta \cos \phi \\ y &= |r| \sin \Theta \sin \phi \\ z &= |r| \cos \Theta \end{aligned} \tag{7-33}$$

whose Jacobian is $(r^2 \sin \Theta)^{-1}$.

The transformation of velocities has to depend on that for coordinates. For if xyz are imagined to be functions of time, then so are $r\Theta\phi$, and the velocities $v_x\ v_y\ v_z = \dot{x}\ \dot{y}\ \dot{z}$ would be related to the time derivatives $\dot{r}\ \dot{\Theta}\ \dot{\phi}$ by means of the following equations (obtained by differentiating Equation (7-33)):

$$v_x = \dot{r} \sin\Theta \cos\phi + \dot{\Theta}\, r \cos\Theta \cos\phi - \dot{\phi}\, r \sin\Theta \sin\phi$$

$$v_y = \dot{r} \sin\Theta \sin\phi + \dot{\Theta}\, r \cos\Theta \sin\phi + \dot{\phi}\, r \sin\Theta \cos\phi$$

$$v_z = \dot{r} \cos\Theta - \dot{\Theta}\, r \sin\Theta \tag{7-34}$$

whose Jacobian is also $(r^2 \sin\Theta)^{-1}$. In the present formulation -- which might be called the hydrodynamical treatment of the two-body problem -- the quantities $\dot{r}\ \dot{\Theta}\ \dot{\phi}$ have to be considered as independent variables. Call them $v_r\ v_\Theta\ v_\phi$. We find

$$v_r = v_x \cos\phi \sin\Theta + v_y \sin\phi \sin\Theta + v_z \cos\Theta$$

$$v_\Theta = \frac{1}{r}\,(v_x \cos\phi \cos\Theta + v_y \sin\phi \cos\Theta - v_z \sin\Theta)$$

$$v_\phi = \frac{1}{r \sin\Theta}\,(-v_x \sin\phi + v_y \cos\phi) \tag{7-35}$$

Now, $G(xyz,\ v_xv_yv_z)$ becomes a <u>new</u> function $\hat{G}(r\Theta\phi v_r v_\Theta v_\phi)$. If we recall the rules

$$\frac{\partial}{\partial x}\,\hat{G}(r(x)\cdots v_r(x_1 v_x)\cdots) = \frac{\partial r}{\partial x}\frac{\partial \hat{G}}{\partial r} + \frac{\partial v_r}{\partial x}\frac{\partial \hat{G}}{\partial v_r} + \cdots \tag{7-36a}$$

and

$$\frac{\partial}{\partial v_x}\,\hat{G} = \frac{\partial v_r}{\partial v_x}\frac{\partial G}{\partial v_r} + \cdots \tag{7-36b}$$

we shall have no trouble converting Equation (7-31) to its new form:

$$\Big[\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + v_\Theta \frac{\partial}{\partial \Theta} + v_\phi \frac{\partial}{\partial \phi} + (r v_\Theta^2 + r \sin^2 \Theta\, v_\phi^2 - \frac{2}{m} X') \frac{\partial}{\partial v_r}$$

$$+ (-\frac{2 v_r v_\Theta}{r} + v_\phi^2 \sin \Theta \cos \Theta) \frac{\partial}{\partial v_\Theta}$$

$$+ (-\frac{2 v_r v_\phi}{r} - 2\, v_\Theta v_\phi \cot \Theta) \frac{\partial}{\partial v_\phi}\Big] \hat{G}$$

$$= \frac{\delta(t-t_o)\,\delta(r-r_o)\,\delta(\Theta-\Theta_o)\,\delta(\phi-\phi_o)\,\delta(v_r-v_r^{\,o})\,\delta(v_\Theta-v_\Theta^{\,o})\,\delta(v_\phi-v_\phi^{\,o})}{(r_o^2 \sin \Theta_o)^2} \tag{7-37}$$

From angular momentum conservation we can deduce three variables $L_x L_y$ and L_z that greatly simplify Equation (7-37). Converting r x v = L to our new coordinates gives

$$L_x = -r^2 (v_\Theta \sin \phi + v_\phi \sin \Theta \cos \Theta \cos \phi)$$

$$L_y = r^2 (v_\Theta \cos \phi - v_\phi \sin \Theta \cos \Theta \sin \phi)$$

$$L_z = r^2\, v_\phi \sin^2 \Theta \tag{7-38}$$

We can use $L_x L_y L_z$ to eliminate v_Θ v_ϕ and Θ $\Big[$not ϕ because $L_x = \frac{\partial}{\partial \phi} L_y$ and $L_y = -\frac{\partial}{\partial \phi} L_x\Big]$. Any function of L_x or L_y or L_z satisfies the homogeneous part of (7-37). We therefore choose a factor of G that will reduce to part of the inhomogeneous term when the remaining variables are correctly given. In fact, we choose

$$G(r\Theta\phi\; v_r v_\Theta v_\phi) = \delta(L_x - L_x^{\,o})\,\delta(L_y - L_y^{\,o})\,\delta(L_z - L_z^{\,o})$$

$$\times\, G(r\phi v_r\; L_x L_y L_z) \tag{7-39}$$

General formulas of use are

$$L^2 = L_x^2 + L_y^2 + L_z^2 = r^4 (v_\theta^2 + v_\phi^2 \sin^2 \theta);$$

$$- L_x \sin \phi + L_y \cos \phi = r^2 v_\theta \; v_\phi = \frac{L_z}{r^2 \sin^2 \theta}$$

$$L^2 - (-L_x \sin \phi + L_y \cos \phi)^2 = r^4 \; v_\phi^2 \sin^2 \theta = \frac{L_z^2}{\sin^2 \theta}$$

From these we conclude that

$$v_\phi = \frac{1}{r^2 L_z} \left[L_z^2 + (L_x \cos \phi + L_y \sin \phi)^2 \right] . \tag{7-40}$$

With the new choice of variables,

$$\left[\frac{\partial}{\partial t} + \left(\frac{L^2}{r^3} - \frac{2X'}{m}\right) \frac{\partial}{\partial v_r} + v_r \frac{\partial}{\partial r} + \frac{L_z^2 + (L_x \cos \phi + L_y \sin \phi)^2}{r^2 L_z} \frac{\partial}{\partial \phi} \right] \bar{G}(r \phi v_r)$$

$$= r_o^2 \sin^2 \phi_o \; \left|\cos 2\theta_o\right| \left|v_\phi^o\right| \delta(t-t_o) \delta(r-r_o) \delta(v_r - v_r^{\,o}) \delta(\phi - \phi_o)$$

$$\tag{7-41}$$

The appearance of $\left|v_\phi^{\,o}\right|$ requires one comment. We see from $L_z = L_z^{\,o}$ that the sign of v_ϕ is a constant -- i.e., the particle always continues in the same angular direction <u>around</u> the center of force. Th_ θdependence could be greatly simplified because, as is well known, the motion of the particle is in a plane fixed in space, determined only by the angular momenta. In fact, if $v_\theta = 0$ and $\theta = \pi/2$ initially, then $v_\theta = 0$ and $\theta = \pi/2$ for all time: i.e., $L_x = L_y = 0$, and the coefficient of $\partial/\partial\phi$ is simply L_z/r^2, the angular velocity in the plane.

Equation (7-41) can be simplified in its r-dependence by use of the constantcy of energy. Apart from a factor of m/2, the reduced mass, the energy is

$$E = \frac{v_r^2}{2} + \frac{1}{2}\frac{L^2}{r^2} + \frac{2X(r)}{m} \tag{7-42}$$

and any function of E satisfies the inhomogeneous part of (7-41). Using

$$v_r = \sqrt{2E - \frac{L^2}{r^2} - \frac{4x^{(r)}}{m}}\ , \tag{7-43}$$

we readily verify that

$$\delta\left[t-t_o - \int_{r_o}^{r} dr'\,\frac{1}{v_{r'}}\right]\frac{1}{|v_r^{\,o}|} \tag{7-44}$$

solves (7-41). If the actual discontinuity comes from a factor $H(t-t_o)$ -- that insures that G is retarded -- then (7-44) gives $\delta(r-r_o)$ exactly as written. Then

$$\delta(E-E_o)|v_r^{\,o}| \tag{7-45}$$

in turn gives

$$\delta(v_r - v_r^{\,o})$$

exactly. Of course the integral in Equation (7-44) must be taken over the complete orbit between $r_o t_o$ and rt, so that v_r in Equation (7-43) must be understood to be the positive root when r is increasing, and the negative root when r is decreasing. Interestingly, the $|v_r^{\,o}|$ of (7-44) and (7-45) cancel. All that is needed to complete the solution is a δ-function of the form

$$\delta \left[\int_{r_o}^{r} \frac{dr'}{g(r')} - \int_{\phi_o}^{\phi} \frac{d\phi'}{f(\phi')} \right] \qquad (7\text{-}46)$$

which satisfies the homogeneous part of Equation (7-41), and reduces to the remaining factors of the inhomogeneous part namely

$$r_o^2 \sin^2 \phi_o \; |\cos 2\theta_o| \; |v_\phi{}^o| \; \delta(\phi - \phi_o)$$

when $r = r_o$. Clearly (7-46) becomes

$$|f(\phi_o)| \quad \delta(\phi-\phi_o) \qquad (7\text{-}47)$$

in this case. Now (7-41) is readily solved by quadratures, and we find that

$$g(r) = r^2 \sqrt{2E - \frac{L^2}{r^2} - \frac{4X}{m}} = r^2 v_r \qquad (7\text{-}48)$$

(again with the appropriate choice of sign) while

$$f(\phi) = \frac{1}{L_z} \left[L_z^2 + (L_x \cos \phi + L_y \sin \phi)^2 \right] \qquad (7\text{-}49)$$

Using (7-38) we find that

$$f(\phi) = r^2 v_\phi \qquad (7\text{-}50)$$

so that the final result is

$$G = H(t-t_o)\,\delta(L_x-L_x{}^o)\,\delta(L_y-L_y{}^o)\,\delta(L_z-L_z{}^o)\,\delta(E-E_o) \;\; x \;\delta\left(t-t_o - \int_{r_o}^{r} \frac{dr'}{v_{r'}}\right)$$

$$x \;\delta\left(\int_{\phi_o}^{\phi} \frac{d\phi'}{f(\phi')} - \int_{r_o}^{r} \frac{dr'}{(r')^2 v_{r'}} \right) \sin^2 \phi_o \; |\cos 2\theta_o| \qquad (7\text{-}51)$$

where

$$H(x) = 1 \qquad \text{if } x > 0$$

$$= 0 \qquad \text{if } x < 0$$

and

$$\frac{\partial}{\partial x} H(x) = \delta(x)$$

This remarkable formula is correct for any two-body central force problem. Let us use it to derive the Boltzmann Equation for a hard-sphere gas.

In a collision between "hard" spheres the interaction acts only for one instant of time, and the pair of spheres does not move any distance during this instant. Actually, no atoms are perfectly hard, but instead may be visualized as having a core of increasing hardness. The only effect of a collision of hard spheres will be to reverse the relative radial velocity, the reversal being the more sudden the "harder" the spheres. When the centers of the spheres, of radius a, are more than 2a apart, the potential between them in zero. At r = 2a the potential is repulsive (positive) and rises steeply to a large value. The simplest analytical representation which captures the physics of the collision process is as follows:

$$X(r) = (2a-r)K \qquad r < 2a$$

$$= 0 \qquad r > 2a \tag{7-52}$$

Hence, the radial force is a constant (<u>large</u>!):

$$F = -X'(r) = K \quad r < 2a$$

$$= 0 \quad r > 2a$$

$$= K \quad H(2a-r) \tag{7-53}$$

where

$$K > 0$$

Let us write out Equation (7-19) with Equation (7-23) substituted to see how this choice for F simplifies the equations

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla\right)_1 f(1t)$$

$$= \frac{1}{m^2} \int\limits_{t_o} \int\limits_2 \int\limits_3 \int\limits_4 F(r_1 - r_2) \cdot (\nabla_{v_1} - \nabla_{v_2}) \; G_{ret}(12t,\ 34t_o)$$

$$\times\ F(r_3 - r_4) \cdot (\nabla_{v_3} \cdot \nabla_{v_4})\, f(3t)\, f(4t_o) \tag{7-54}$$

$$- \frac{1}{m} \int\limits_2 F(r_1 - r_2) \cdot (\nabla_{v_1} - \nabla_{v_2})\, f(2t)\, f(1t)$$

$$= \frac{4K^2}{m^2} \int\limits_2 \int\limits_{R_o} \int\limits_{V_o} \int\limits_{v_o} \int\limits_{r_o} \int\limits_{t_o} H(2a-r)\ H(2a-r_o)$$

$$\frac{\partial}{\partial v_r} G_{ret} \frac{\partial}{\partial v_{r_o}} ff - \frac{2K}{m} \int\limits_2 H(2a-r) \frac{\partial}{\partial v_r} \ ff \tag{7-55}$$

Making use of the fact that K is very large (compared to other forces, e.g., the centrifugal forces) we can simplify Equation (7-51). In particular, in $\delta(t-\cdots)$ let us replace dr by dv_r. Using $E=E_o$, we find that

$$\frac{dr}{v_r} = \frac{dv_r}{\frac{2K}{m} + \frac{L^2}{r^3}} \sim dv_r \, \frac{m}{2K}$$

when K is sufficiently large. Hence,

$$\delta(t-t_o - \int \frac{dr}{v_r})\,\delta(E-E_o) \longrightarrow \frac{m}{2K}\,\delta(t-t_o - \frac{m}{2K}(v_r-v_r{}^o)\,\delta(r-r_o - \frac{m}{4K}\,(v_r^2 - v_{r_o}^2)$$

(7-56)

Because of the first factor of (7-56), we can replace $H(t-t_o)$ in (7-51) by $H(v_r-v_r{}^o)$. Equation (7-56) shows that as $K\to\infty$, $t\to t_o$, $r\to r_o$, so that in (7-51), $\phi\to\phi_o$, $v_\theta\to v_\theta{}^o$, $v_\phi\to v_\phi{}^o$ and the net constant is unity!

Upon substituting G into Equation (7-55), the term resulting from $\partial/\partial v_r\ H(v_r v_r{}^o) = \delta(v_r-v_{r_o})$ exactly cancels the 2nd term on the right-hand side of (7-46). The remaining terms must be evaluated in the $K\to\infty$ limit. Several of the integrals may be performed directly, leaving the structure

$$I = \frac{2K}{m} \int_2 \int_{t_o} \int_o^{2a} dr_o \int_{-\infty}^{\infty} dv_{r_o}\ H(v_r-v_{r_o})\ \frac{\partial}{\partial v_r} \left\{\delta\left[r-r_o - \frac{m}{4K}\,(v_r^2 - v_{r_o}^2)\right]\right.$$

$$\left. \times\ \delta\left[t-t_o - \frac{m}{2K}\,(v_r-v_{r_o})\right]\right\} \frac{\partial}{\partial v_{r_o}}\ ff\ \left[r_o\ \theta\phi,\ v_{r_o}\ v_\theta v_\phi, V, R+V(t-t_o), t_o\right]$$

(7-57)

Now

$$\int_o^{2a} dr_o\,\delta\ \left[r-r_o - \frac{m}{4K}\,(v_r^2 - v_{r_o}^2)\right] = 1$$

if $r - \frac{m}{4K}(v_r^2 - v_{r_o}^2)$ is less than 2a, = 0 otherwise. Hence, we can write

$$H\left[2a - r + \frac{m}{4K}(v_r^2 - v_{r_o}^2)\right]$$

for it. Performing also the t_o integral leads us to

$$I = \frac{2K}{m} \int_2 \int_{-\infty}^{\infty} dv_{r_o}\, H(v_r - v_{r_o}) \frac{\partial}{\partial v_r}$$

$$\times H\left[2a - r + \frac{m}{4K}(v_r^2 - v_{r_o}^2)\right] \frac{\partial}{\partial v_{r_o}} \text{ ff}$$

$$\times \left[r - \frac{m}{4K}(v_r^2 - v_{r_o}^2), \theta\phi;\ v_{r_o} v_\theta v_\phi,\ V, R + V\frac{m}{2K}(v_r - v_{r_o}), t + \frac{m}{2K}(v_r - v_{r_o}\right.$$

(7-58)

Now in this equation $\partial/\partial v_{r_o}$ must be understood to act only on the v_{r_o} dependence that existed in ff <u>before</u> we did the t_o integral; but $\partial/\partial v_r$ acts on the v_r dependence in H and ff. To complete the derivation, we have to note that the $K \to \infty$ limit may be performed in ff before $\partial/\partial v_r$ or $\partial/\partial v_{r_o}$ are applied, but not in H. The reason for this is that ff will be a smoothly varying function of its arguments, whereas H's properties change discontinuously when it loses its v dependence! So, set $K = \infty$ in ff, and evaluate

$$\frac{2K}{m}\frac{\partial}{\partial v_r} H\left[2a - r + \frac{m}{4K}(v_r^2 - v_{r_o}^2)\right] = v_r \frac{\partial}{\partial\left(\frac{m}{4K} v_r^2\right)} \cdot H$$

$$= v_r\, \delta\left[2a - r + \frac{m}{4K}(v_r^2 - v_{r_o}^2)\right] \qquad (7\text{-}59)$$

Since $r < 2a$, (7-59) requires the v_{r_o} integral to be restricted by the condition $v_{r_o}^2 > v_r^2$. Subject to this restriction, however, (7-59) may be replaced by its limiting value

$$v_r \delta(r-2a)$$

Since $H(v_r - v_{r_o})$ implies $v_r > v_{r_o}$, the v_{r_o} integral extends to v_r when v_r is negative, and $-v_r$ when v_r is positive. Thus, in either case of the v_{r_o} integral

$$\left(\frac{\partial}{\partial t} + v_1 \cdot \nabla_1\right) f(r_1 v_1 t)$$

$$= \int_{r_2} \int_{v_2} v_r \, \delta(r-2a) \; ff \left[r\theta\phi, \; - v_r v_\theta v_\phi, \; V, R, t\right] \qquad (7\text{-}60)$$

where

$$V = \frac{v_1+v_2}{2} \qquad v = v_1 - v_2$$

$$R = \frac{r_1+r_2}{2} \qquad r = r_1 - r_2$$

and

$$ff = f(R + \tfrac{1}{2}\, r, \; V + \tfrac{v^*}{2}) \;\; f(R - \tfrac{1}{2}\, r, \; V - \tfrac{v^*}{2})$$

with

$$v^* = -\left|v_r\right| \;\; v_\theta v_\phi \qquad (7\text{-}61)$$

The collision term has a clear meaning. The $v_r > 0$ and $v_r < 0$ contributions determine respectively the "IN" and "OUT" contributions.

The rate at which particles are scattered out of the r_1v_1 component of f involves the velocities v_1 and v_2 before collision; but for particles scattered <u>into</u> the r_1v_1 component depends on the velocities v_1' and v_2' that must exist before collision in order that v_1 and v_2 occur afterwards! Now $R + 1/2\ r = r_1$ and $R - 1/2\ r = r_2$, and

$$V + \frac{v^*}{2} = v_1$$

$$V - \frac{v^*}{2} = v_2 \qquad \text{when } v_r < 0$$

$$V + \frac{v^*}{2} = v_1 - e(e \cdot v)$$

$$V - \frac{v^*}{2} = v_2 + e(e \cdot v) \qquad \text{when } v_r > 0$$

where e is the unit vector joining the centers of the colliding hard spheres.

One can easily verify from these equations that the conservation of mass flow still holds. The methods which are usually applied to the Boltzmann equation are thoroughly described in Chapman and Cowling's work on "The Mathematical Theory of Nonuniform Gases" (Cambridge Press, second edition, 1952). We shall consider an attempt at deriving directly from the <u>linearized equation</u> the dispersion relation describing the possible modes of the system. At a later date the same procedure will be considered for the more interesting problem of the plasma.

CHAPTER VIII

LINEARIZATION OF THE BOLTZMANN EQUATION FOR HARD SPHERES

Suppose that a gas of hard-sphere-like particles, at a finite temperature, is sharply disturbed so that a small group of particles is set moving in a definite direction with high speeds. In fact, suppose that a signal is communicated to the gas by varying the amplitude of the disturbances with time. The particles so excited represent a certain amount of momentum and energy injected into the gas which, if the gas is isolated, will be conserved throughout the future history of the whole gas. When the density is low, so that the mean-free-path is considerable, the disturbance-signal will be found across the gas with very little change. When, however, collisions come into play quickly, the energy and momentum gets shared over larger and larger groups of particles. The limiting situation may be described as follows: the final motion of the gas as a whole resembles the gas motion before disturbance, but with a slightly higher average energy per particle (a higher gas temperature). The average momentum is lost to the vessel containing the gas. No signal is received across the gas if this complete thermalization takes place in times less than <u>the least time possible</u> for a remnant of the original signal to get through in any recognizable shape.

Now, except for the very low density case, the original signal will be thermalized locally; that is, the impulse of energy and momentum rapidly will be distributed to a larger group of particles, though not to the whole gas. In this state of its evolution, the impulse will consist of a small region of the gas of slightly higher temperature than the rest of the gas, and of small nonzero average momentum compared to the rest of the gas. Whether any part of the signal will be detected <u>after this stage</u> has been reached can be studied by treating solutions of the gas equation which differ only slightly from the equilibrium solution. This approach permits us to linearize the gas equation (in this case, Boltzmann's equation) and simplifies the ensuing analysis.

That such a treatment must -- at a finite temperature -- lead to the possibility of transmission of signals is known from direct experience with the phenomena of sound, and also from the possibility of deriving the sound velocity from the linearized form of Euler's Equations, together with the thermodynamic equation of state for an <u>ideal gas</u>. By an "Ideal" gas we mean one without any important interaction <u>except</u> that its particles must be capable of interchanging energy and momentum to bring about an equilibrium velocity distribution. (Our hard-sphere gas approximates the ideal gas in a reasonable way.) The actual calculation is simple. The time derivative of the mass continuity equation reads

$$\frac{\partial^2 \rho}{\partial t} = -\nabla \cdot \frac{\partial}{\partial t} j \tag{8-1}$$

The linearized Euler Eq. relates $\partial/\partial t\ j$ to the pressure, p.

$$\frac{\partial}{\partial t} j = -\nabla p \tag{8-2}$$

The equilibrium equation of state, for an ideal mono-atomic gas, is

$$p = KT \frac{N}{V} = \frac{KT}{m} \rho \tag{8-3}$$

But adiabatic variations of the pressure follow the familiar law

$$\frac{\Delta p}{p} = \frac{c_p}{c_v} \frac{\Delta(1/v)}{(1/v)} = \frac{5}{3} \frac{\Delta\rho}{\rho}$$

Thus, the density fluctuations are given by

$$\frac{\partial^2}{\nabla t^2} \rho = \frac{5}{3} \frac{KT}{m} \nabla^2 \rho \tag{8-4}$$

which has the dispersion relation

$$\omega^2 = v_s^2 \ k^2 \tag{8-5}$$

with

$$v_s = \sqrt{\frac{5}{3}\frac{KT}{m}} \tag{8-6}$$

v_s is also one definition of the average thermal velocity. The problem we set for ourselves is to derive the dispersion relations which follow from the hard-sphere equation; to investigate the way particles of finite size contribute damping terms to equations like Eq. (8-5). This means, in effect, deriving the thermodynamic equations of state from first principles. That this can be done to some extent was shown by an analysis of Wang, Chang, and Uhlenbeck, an outline of the method being given in lecture notes of Uhlenbeck.(8) We shall take that method as a guideline for our researches into the microscopic derivation of macroscopic wave-motions. In the present Chapter we shall give a brief treatment of it.

We have derived the Boltzmann Eq. in the preceding section. We transcribe it here in a new form by letting

$$e = \frac{r_1 - r_2}{|r_1 - r_2|}$$

$$v_r = \frac{(r_1 - r_2)}{|r_1 - r_2|} \cdot (v_1 - v_2) = e \cdot (v_1 - v_2)$$

and

$$v_1' = v_1 - v_r \ e$$

$$v_2' = v_2 + v_r \ e$$

The result is

$$(\frac{\partial}{\partial t} + v_1 \cdot \nabla_1) f(r_1 v_1 t) = \int_{r_2} \int_{v_2} v_r \, \delta(|r_1 - r_2| - 2a)$$

$$x \begin{cases} f(r_1 v_1 t) f(r_2 v_2 t) & v_r < 0 \\ f(r_1 v_1{}' t) f(r_2 v_2{}' t) & v_r > 0 \end{cases} \tag{8-7}$$

We shall take the step of making the hard-sphere gas into an ideal gas. This can be done by imagining the radius a of the spheres to be so small that we can replace r_2 by r_1 in f because of $\delta(|r_1-r_2|-2a)$. The direction of the vector $r_1 - r_2$, or of e, is still quite important, however. After setting in $f(r_2)$ $r_2 \sim r_1$ we can then introduce instead of r_1 and r_2 the variables $r = r_1$ and $\bar{r} = r_1 - r_2$. The integral over $\bar{r}$ gives $(2a)^2$ and an angular integral $\int d\Omega_e$ over the directions of e. This leads to

$$(\frac{\partial}{\partial t} + v_1 \cdot \nabla) f(r v_1 t) = (2a)^2 \int d\Omega_e \int (dv_2) v_r \begin{cases} f(r v_1 t) f(r v_2 t) & v_r<0 \\ f(r v_1' t) f(r v_2' t) & v_r>0 \end{cases} \tag{8-8}$$

The cumbersome restriction on v_r can be removed now by replacing v_r by $\pm v_r$ where indicated, and extending the integrals over the whole range with an extra factor of 1/2.

$$(\frac{\partial}{\partial t} + v_1 \cdot \nabla) f(r v_1 t) = 1/2(2a)^2 \int_e d\Omega \int_{v_2} dv_2 \, |e \cdot (v_1 - v_2)| \Big\{ f(r v_1' t) f(r v_2' t) - f(r v_1 t) f(r v_2 t) \Big\} \tag{8-9}$$

We linearize Eq. (8-9) by setting

$$f(rvt) = n_o f_o(v)(1 + g(rvt)) \qquad (8\text{-}10)$$

where n_o is the density of particles N/V, and $f_o(v)$ is the normalized Maxwell-Boltzmann function:

$$f_o(v) = \left(\frac{\alpha}{\pi}\right)^{3/2} e^{-\alpha v^2}$$

$$\alpha = m/2KT$$

$$\int (d^3v) f_o(v) = 1 \qquad (8\text{-}11)$$

We neglect terms involving products of two g's. The terms without g are

$$\int_e d\Omega \int_{v_2} dv_2 \; e\cdot(v_1-v_2) \left\{ f_o(v_1') f_o(v_2') - f_o(v_1) f_o(v_2) \right\} , \qquad (8\text{-}12$$

which vanishes identically. For

$$(v_1')^2 = v_1^2 - 2\, v_r(v_1\cdot e) + v_r^2$$

$$(v_2')^2 = v_2^2 + 2\, v_r(v_2\cdot e) + v_r^2$$

and

$$(v_1')^2 + (v_2')^2 = v_1^2 + v_2^2 \qquad (8\text{-}13)$$

since

$$(v_1 - v_2)\cdot e = v_r \;.$$

Hence,

$$f_o(v_1')f_o(v_2') = f_o(v_1)f_o(v_2) \quad . \tag{8-14}$$

Therefore,

$$(\frac{\partial}{\partial t} + v_1 \cdot \nabla)g(rv_1t) = \frac{n_o}{2}(2a)^2 \int_e d\Omega \int_{v_2} dv_2 \left|e\cdot(v_1-v_2)\right| f_o(v_2)$$

$$x \left\{g(rv_1't) + g(rv_2't) - g(rv_1t) - g(rv_2t)\right\} \tag{8-15}$$

in which we have used (8-14) again. The r and t transforms of g(rtv), which we call $G_{k\omega}(v)$, satisfies:

$$(\omega - v_1\cdot k)G(v_1) = i\,\frac{n_o}{2}(2a)^2 \int_e d\Omega \int_{v_2} dv_2 \left|e\cdot(v_1-v_2)\right| f_o(v_2)$$

$$x \left\{G(v_1') + G(v_2') - G(v_1) - G(v_2)\right\} \tag{8-16}$$

or

$$(\omega - v_1\cdot k)G(v) = i\,J\ \ G(v) \tag{8-17}$$

in which J is the linear integral operator acting on G.

There is a formal approach to (8-17) that is very suggestive. Let us suppose that there exist orthonormal eigenfunctions of J, and let them be ψ_n with λ_n the eigenvalues. Then

$$J\psi_n(v) = \lambda_n\psi_n(v)$$

where

$$\int (dv)\psi_n(v)\psi_m(v)f_o(v) = \delta_{mn} \tag{8-18}$$

There are five functions of velocity that automatically make the collision term vanish. They are, apart from constants and orthogonalization,

$g = 1$, a constant

$g = \vec{v}$, the three components of velocity

$g = v^2$, the kinetic energy.

Hence, there are five zero-eigenvalue eigenfunctions of J. The remaining eigenvalues are negative. For

$$\lambda_i = \frac{\int (dv_1)\, f_o(v_1)\psi_i(v_1)\, J\left[(\psi_i(v_1)\right]}{\int (dv_1)\, f_o \psi_i^2}$$

and the numerator can be written

$$\int (dv)\, f_o \psi_i J(\psi_i) = -1/4\, \frac{n_o}{2}(2a) \int_e d\Omega \int dv_1 \int dv_2 \left|e\cdot(v_1 - v_2)\right| f_o(v_1) f_o(v_2)$$

$$\times \left\{\psi_i(v_1') + \psi_i(v_2') - \psi_i(v_1) - \psi_i(v_2)\right\}^2$$

We get this result by first interchanging v_1 and v_2, then interchanging $v_1 v_2$ and $v_1' v_2'$, and taking the average of the four results.

That most of the eigenvalues are negative means only that the collision term causes damping of most disturbances.

The only disturbances that the collision term cannot influence are changes in density (1), momentum (v) and energy (v^2), i.e., temperature, of the gas.

If we knew the orthonormal eigenfunctions, and if we knew that they formed a complete set, we could expand an arbitrary G(v) in a series:

$$G(v) = \sum_m c_m(k\omega)\psi_m(v) \tag{8-19}$$

Then (8-17) becomes

$$\sum_m c_m(\omega - v\cdot k - i\lambda_m)\psi_m(v) = 0 \tag{8-20}$$

or

$$(\omega + i\lambda_m)c_m - k \cdot \sum_n v_{mn}c_n = 0 \tag{8-21}$$

where

$$v_{mn} = \int (d^3v)\psi_m(v)v\psi_n(v)f_o(v) \tag{8-22}$$

is a matrix element of v.

The dispersion relation for wave motiones in the gas are given by the condition that (8-21) have a solution:

$$\det(\omega + i\lambda_m)\delta_{mn} - k \cdot v_{mn} = 0 \tag{8-23}$$

The problem of the completeness of the ψ_m has not been solved, neither in general nor for hard-sphere collision terms. Nor is it even known that the spectrum of λ's is discrete, although both completeness and discreteness are entirely plausible. Still, Eq. (8-23) is very revealing. Let us restrict ourselves only to the five eigenfunctions for which $\lambda = 0$. If we solve (8-23) for this very restricted subspace of the full function space of the ψ_m, we find -- with Wang, Chang and Uhlenbeck -- the dispersion relation for sound waves.

Since $f_o(v)$ is normalized, $\psi_1 = 1$ is also normalized (relative to $f_o(v)$). ψ_2, ψ_3 and ψ_4 are porportional to v_x v_y v_z, respectively. They are therefore orthogonal to ψ_1. For normalization we find

$$\psi_v = \sqrt{2\alpha}\ \vec{v}$$

Now v^2 is not orthogonal to ψ_1, but $v^2 - \frac{3}{2\alpha}$ is. The normalized function is

$$\psi_5 = \alpha \sqrt{2/3}\ (v^2 - \frac{3}{2\alpha})$$

The matrix elements of v_x are

$$(v_x)_{mn} = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & \frac{1}{\sqrt{2\alpha}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2\alpha}} & 0 & 0 & 0 & \frac{1}{\sqrt{3\alpha}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3\alpha}} & 0 & 0 & 0 \end{pmatrix} \end{array}$$

If $k_y = k_z = 0$, we get

$$\begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} \omega & \frac{-k_x}{\sqrt{2\alpha}} & 0 & 0 & 0 \\ \frac{-k_x}{\sqrt{2\alpha}} & \omega & 0 & 0 & \frac{-k_x}{\sqrt{3\alpha}} \\ 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 \\ 0 & \frac{-k_x}{\sqrt{3\alpha}} & 0 & 0 & \omega \end{pmatrix} \end{array} = 0 \qquad (8\text{-}24)$$

Or

$$\omega^3(\omega^2 - \frac{5k_x^2}{6\alpha}) = 0$$

But $\alpha = m/2KT$, so that

$$\omega^2 = \frac{5}{3}\frac{KT}{m}k_x^2$$

By symmetry,

$$\omega^2 = \frac{5}{3}\frac{KT}{m}k^2 \qquad (8\text{-}25)$$

Thus, there is every hope that a purely microscopic approach to the problem of wave-motions in a multi-particle system will succeed, and without undue analytical complications. The alternate approach, the Macroscopic methods are best suited to the formulation of boundary conditions when such conditions are easier to obtain empirically than theoretically.

To continue with the solution of the linearized Boltzmann equation, let us note that the nonzero collision terms are numerically very important. Returning to Eq. (8-15), we can estimate the magnitude of a typical term, the third on the right-hand-side. Introducing transforms, bringing it to the left side, and replacing v_1-v_2 by v' and v_1 by v, we get

$$\omega - v \cdot k + \frac{i}{2} n_o(2a)^2 \int_{v'} dv' \int_{e} d\Omega \; |e\cdot v'| f_o(v-v')G(v) = \cdots$$

Now

$$\int d\Omega_e \; |e\cdot v'| = 2\pi|v'|$$

as can be seen by letting v' be along the polar axis of the e-coordinates in a spherical system. At zero temperature

$f_o(v-v') = \delta^{(3)}(v-v')$, and the complex factor contains the frequency

$$\nu = \pi n_o (2a)^2 |v| \tag{8-26}$$

Now, we do not expect a sound-wave at zero temperature, but the quantity of (8-26) will be typically important as a measure of the collision rates. To get an idea of the difficulties that are in store in any microscopic treatment of gases, consider the magnitude of ν for air at room temperature. To sufficient approximation we can treat a nitrogen (N_2) gas, let $2a \sim 3.5 \times 10^{-8}$ cm be a typical atomic diamter, $m \sim 43 \times 10^{-24}$ gms per molecules; a density of 0.00129 gm/cm^3 leads to $n_o \sim 3 \times 10^{19}$ molecules/cm^3; a temperature of 300^{o}K leads to a thermal velocity of 4.6×10^4 cm/sec; so that a thermal molecule experiences a collision frequency

$$\nu \sim 4 \times 10^9/\text{sec} \tag{8-27}$$

With such a value, how can a sound wave of 10^3 cps frequency travel more than about

$$d \sim \frac{v_{th}}{\nu} \sim 10^{-5} \text{ cm} \tag{8-28}$$

For ν in $G(v)$ would completely dominate ω, in fact wiping out all reference to ω! For this not to happen in practice, the remaining terms in (8-13) must cancel off most of this damping effect. What happens in practice is this: a disturbance of the air, $g(rvt)$, can be thought of as the superposition of characteristic disturbances, or "eigen-disturbances". These include changes that leave a permanent alteration in the density, momentum and energy of the air ($\lambda = 0$), together with all other possible changes in the local velocity distribution, which are transient ($\lambda \neq 0$). The latter are, in fact, rapidly damped, but the former survive as long as the conservation laws permit. A locally produced sound dies away with distance, since the "density-momentum-energy" is shared over larger and larger surfaces with time. But an ideal plane-wave remains intact because each element of the advancing wave front receives equal contributions of "density-momentum-energy" from all sides.

Sound waves will exist in a plasma also, as can be understood from the great generality of the processes involved. But more often than not the excitations of the plasma that are of most interest are those for which $\lambda \neq 0$, and for which a knowledge of λ_i (and ψ_i) are of the greatest importance. In this report we have dealt with hydromagnetic non-damped wave-motions derived from a model of the plasma which ignored collisions entirely. That we were justified can be argued only by showing that indeed collisions are unimportant (at zero temperature). The sequel to this report will consider this question. We can, however, note briefly that unlike the hard-sphere case, for which a collision frequency exists for a particle of non-zero velocity in a gas at zero temperature, there is no intrinsic radius of repulsion for electrons and ions. In fact, the long-range forces, and the mixture of repulsion and attraction in the plasma, produces a variety of collective phenomena which the hard-sphere model can in no way duplicate. The search for solutions to the problem of the wave-motions in a plasma promises great rewards.

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2. Alfven velocity
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